

Union-Closed vs Upward-Closed Families of Finite Sets

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Abstract

A finite family \mathcal{F} of subsets of a finite set X is union-closed whenever $f, g \in \mathcal{F}$ implies $f \cup g \in \mathcal{F}$. These families are well known because of Frankl's conjecture [10]. In this paper we developed further the connection between union-closed families and upward-closed families started in [18] using rising operators. With these techniques we are able to obtain tight lower bounds to the average of the length of the elements of \mathcal{F} and to prove that the number of joint-irreducible elements of \mathcal{F} can not exceed $2\binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}$ where $|X| = n$.

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1 Introduction

Consider a finite set $X = \{a_1, \dots, a_n\}$ formed by $n \geq 1$ elements. A family of subsets \mathcal{F} of the powerset 2^X such that for any $f, g \in \mathcal{F}$, $f \cup g \in \mathcal{F}$ is called union-closed (briefly \cup -closed). Without loss of generality we can assume that $X = \bigcup_{f \in \mathcal{F}} f$ and for the rest of the paper, when it is not differently stated, \mathcal{F} will denote a \cup -closed family on $X = \{a_1, \dots, a_n\}$ with $X = \bigcup_{f \in \mathcal{F}} f$. In 1979, Frankl stated the following conjecture

Conjecture 1. *For all union-closed families \mathcal{F} , there exists an $a \in X$ such that $|\{f \in \mathcal{F} : a \in f\}| \geq \frac{|\mathcal{F}|}{2}$.*

Although many attempts to solve this simple-sounding conjecture have been made, this remains open and has become known as the *union-closed conjecture* or *Frankl's conjecture*. A simple argument in [14] shows that there is an $a \in X$ which is contained in at least $|\mathcal{F}|/\log_2(|\mathcal{F}|)$ elements of \mathcal{F} . In [22], this bound is improved by a multiplicative constant. The conjecture holds if $|\mathcal{F}| < 40$ (see [15, 20]) or $|X| \leq 11$ (see [6, 16]) or $|\mathcal{F}| > \frac{5}{8}2^{|X|}$ (see [7, 8, 9]) or \mathcal{F} contains some collection of small sets (see [6, 16]).

The family \mathcal{F} is a semilattice with respect to the union operation, furthermore, since \mathcal{F} is finite we can endow $\mathcal{F} \cup \{\{\emptyset\}\}$ with a structure of lattice. In this direction it is possible to give another formulation of Frankl's conjecture in the framework of lattice theory. Let (L, \vee, \wedge) be a finite lattice, we denote by $J(L)$ the set of *join-irreducible* elements, i.e., the elements $z \in L$ such that if $g = x \vee y$ then $x = z$ or $y = z$. Denoting by $V_x = \{y \in L : y \leq x\}$ the principal filter generated by x , Frankl's conjecture is equivalent to the following lattice theoretic conjecture

Conjecture 2.

$$\frac{1}{|L|} \min\{|V_x| : x \in J(L)\} \leq \frac{1}{2}.$$

This approach has received a significant amount of attention (see [1, 2, 3, 4, 7, 8, 9, 11, 17, 19, 21]). Although much research has been done on union-closed families, it seems there is no general tool to tackle this problem. In a different direction, Reimer in [18] developed a connection between \cup -closed sets and upward-closed sets by using a repeated application of rising

operators. From this connection he provided a lower bound on the average of the length of the elements of \mathcal{F} showing

$$\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} |f| \geq \frac{\log_2(|\mathcal{F}|)}{2}$$

The aim of this paper is to develop further the correspondence introduced by Reimer and study more deeply the consequences and some of the results that can be achieved from this point of view. The hope is to give an approach to the study of \cup -closed families of sets that can be helpful to give some insight to a possible solution of the Conjecture 1.

The paper is organized as following: in Section 2 we give some definitions and we fix the notation, in Sections 3 and 4 we extend the results of [18], in Section 5 we use this approach to obtain some lower bounds on the localized average of the length of the elements of \mathcal{F} . More precisely, given $S \subseteq \mathcal{F}$, we provide lower bounds to the quantity

$$\frac{1}{|\{f \in \mathcal{F} : \exists z \in S, z \subseteq f\}|} \sum_{f \in \mathcal{F} : \exists z \in S, z \subseteq f} |f|$$

Finally in Section 6 we use these techniques to prove that the number of joint-irreducible elements of $\mathcal{F} \subseteq 2^X$ is at most $2\binom{n}{\lfloor n/2 \rfloor} + \binom{n}{\lfloor n/2 \rfloor + 1}$ where $|X| = n$.

2 Preliminaries

For an element $t \in 2^X$ we denote the cardinality of t by $|t|$. Let us fix a subset $S \subseteq 2^X$ (note that the cardinality of S is also denoted by $|S|$). Let $a \in X$, S is partitioned into two subsets $S_a, S_{\bar{a}}$ of the elements of S containing a , not containing a , respectively. We can see S endowed with the order induced by the relation \subseteq as a poset (S, \subseteq) , thus an *antichain* $A \subseteq S$ is a non-empty subset such that any pair of elements of A is incomparable. We denote the set of minimal (maximal) elements of S by $\min(S)$ ($\max(S)$). Both $\min(S), \max(S)$ are clearly antichains. Let us denote the set of all f^c for $f \in S$, where $f^c = X \setminus f$ is the complement set of f , by S^c . Given two different elements $f, g \in S$ we say that g *covers* f , written $f < g$ if there is no $h \in S$ such that $f \subsetneq h \subsetneq g$.

An *upward-closed family* (also called upset or filter, see [5]) is a subset $\mathcal{F} \subseteq 2^X$ such that if $f \subseteq g$ for some $f \in \mathcal{F}, g \in 2^X$, then $g \in \mathcal{F}$, and a *downward-closed family* (also downset or simplicial complex) is defined

analogously. Note that an upward-closed family is a \cup -closed set.

Let us consider a \cup -closed family \mathcal{F} of 2^X . An ideal of \mathcal{F} is a subset $I \subseteq \mathcal{F}$ such that $I \cup g \subseteq I$ for all $g \in \mathcal{F}$. Let $z \in 2^X$, the *principal ideal of z* denoted by $\mathcal{F}[z]$, is the (possibly empty) set of all the elements of \mathcal{F} containing z . Clearly if $z \in \mathcal{F}$, then $\mathcal{F}[z]$ is the principal ideal generated by z , i.e., $\mathcal{F}[z] = z \cup \mathcal{F} = \{z \cup f, f \in \mathcal{F}\}$. This definition can be extended to any sub-family $S \subseteq \mathcal{F}$, thus the *principal ideal generated by S* is the set $\mathcal{F}[S] = \cup_{z \in S} \mathcal{F}[z]$. It is straightforward to see that $\mathcal{F}[\min(\mathcal{F})] = \mathcal{F}$. In the particular case $\mathcal{F} = 2^X$, we use the shorter notation S^\uparrow for the set $2^X[S]$.

An element $g \in \mathcal{F}$ is called *irreducible* whenever $g = h \cup t$ implies $h = g$ or $t = g$. We denote by $J(\mathcal{F})$ the set of irreducible elements of \mathcal{F} . It is evident that $\min(\mathcal{F}) \subseteq J(\mathcal{F})$. This set plays an important role since it is the minimal set of generators of the semilattice (\mathcal{F}, \cup) . This set is \cup -*independent*, in the following sense. Given $S \subseteq 2^X$, we say that S is \cup -independent whenever no element $z \in S$ can be written as a union of elements in $S \setminus \{z\}$.

3 Union-closed and upward-closed families

In this section we further explore the connection between \cup -closed families and upward-closed families. This connection has been already established in [18] using the concept of *rising function*, a well-known operator used also by Frankl in [13]. We briefly recall such operator. Given an element $a \in X$ and a family (not necessarily \cup -closed) of subsets $S \subseteq 2^X$, the rising function $\varphi_{S,a} : S \rightarrow 2^X$ is the function defined for all $z \in S$ by

$$\varphi_{S,a}(z) = \begin{cases} z \cup \{a\} & \text{if } z \cup \{a\} \notin S, \\ z & \text{otherwise;} \end{cases}$$

This is a one-to-one function $\varphi_{S,a} : S \rightarrow 2^X$ and the image $\varphi_{S,a}(S)$ is called the *a -rising of S* . In [18], the author iterates these rising functions in the following way. Let $X = \{a_1, \dots, a_n\}$ and let φ_0 be the identity function on 2^X , and $S_0 = S$, then for all $1 \leq j \leq n$ let

$$S_j = \varphi_j(S_{j-1}), \quad \varphi_j = \varphi_{S_{j-1}, a_j} \circ \varphi_{j-1}$$

We call φ_n the *rising function with respect to the word $w = a_1 a_2 \dots a_n$* of the family S and we denote it by φ_w to underline the dependency of this map from the order used to perform these iterations. We call each S_i the

i -section and for any $z \in S$ the elements $z_i = \varphi_i(z)$ for $i = 0, \dots, n$ is called the *trajectory* of z through the iterated application of the rising functions. We immediately note that this definition depends on the order in which the rising functions are iterated. Indeed consider the set $X = \{a, b, c\}$ and the \cup -closed family $\mathcal{F} = \{\{a\}, \{a, b, c\}\}$, it is evident that $\varphi_w(\mathcal{F}) \neq \varphi_{w'}(\mathcal{F})$ when $w = abc, w' = acb$. We denote by \mathfrak{S}_X the permutation group of the set of objects $X = \{a_1, \dots, a_n\}$, and for a word $w = a_1 \dots a_n$, we use the notation $w\theta = \theta(a_1) \dots \theta(a_n)$. There is an evident action of \mathfrak{S}_X on the set $\{\varphi_{w\vartheta}(g) : g \in \mathcal{F}, \vartheta \in \mathfrak{S}_X\}$ given by $\sigma \cdot \varphi_{w\vartheta}(g) = \varphi_{w\vartheta\sigma}$. In Section 4 we characterize the orbits of such action and we explore some consequences. It is not difficult to see that the rising function φ_w is a bijection between the family S and its image $\varphi_w(S)$, moreover from the definition it is easy to verify that, independently from the condition that S is \cup -closed, $\varphi_w(S)$ is upward-closed. We have the following lemma:

Lemma 1. *With the notation above, if there are two different elements $z, z' \in S$ such that $\varphi_i(z) = \varphi_i(z') \cup \{a_{i+1}\}$, then $a_{i+1} \in z \setminus \varphi_w(z')$.*

Proof. If $a_{i+1} \notin z$, then $a_{i+1} \notin \varphi_k(z)$ for all $k \leq i$, but this contradicts $\varphi_i(z) = \varphi_i(z') \cup \{a_{i+1}\}$, thus $a_{i+1} \in z$. Since φ_i is a bijection and $\varphi_i(z) = \varphi_i(z') \cup \{a_{i+1}\}$ with $z \neq z'$, we get $a_{i+1} \notin \varphi_i(z')$ and so $a_{i+1} \notin \varphi_w(z')$. \square

If we add the \cup -closed condition, we have the following lemma:

Lemma 2. [18] *Let \mathcal{F} be a \cup -closed family of subsets of $X = \{a_1, \dots, a_n\}$. For each $0 \leq i \leq n$ the i -section \mathcal{F}_i is a \cup -closed family.*

The following lemma is a consequence of [18, Lemma 3.3], but for the sake of completeness we present here with proof.

Lemma 3. *Let \mathcal{F} be a \cup -closed family of sets and let $f \in \mathcal{F}$. Consider the rising function φ_w with respect to the word $w = a_1 a_2 \dots a_n$ of the family \mathcal{F} . Then if t belongs to the i -section \mathcal{F}_i , for some $1 \leq i \leq n$, then also $t \cup f \in \mathcal{F}_i$.*

Proof. We prove it by induction on the index i . Suppose that $i = 0$, since $\mathcal{F}_0 = \mathcal{F}$ is a \cup -closed family, if $t \in \mathcal{F}$ then, since $f \in \mathcal{F}$, $t \cup f \in \mathcal{F} = \mathcal{F}_0$. Suppose that the statement of the theorem is true for i and let us prove it for $i + 1$. Suppose $t \in \mathcal{F}_{i+1}$ and let $\bar{t} = \varphi_{\mathcal{F}_i, a_{i+1}}^{-1}(t) \in \mathcal{F}_i$. By the inductive hypothesis $\bar{t} \cup f \in \mathcal{F}_i$, we consider several cases.

Case 1. Suppose $a_{i+1} \in \bar{t}$. Thus $a_{i+1} \in \bar{t} \cup f \in \mathcal{F}_i$, hence $\bar{t} = \varphi_{\mathcal{F}_i, a_{i+1}}(\bar{t}) = t$ and $\varphi_{\mathcal{F}_i, a_{i+1}}(\bar{t} \cup f) = \bar{t} \cup f = t \cup f$ and so $t \cup f \in \mathcal{F}_{i+1}$.

Case 2. Suppose $a_{i+1} \notin \bar{t}$. We consider two further subcases.

- $a_{i+1} \in t$, hence necessarily by definition of the rising function $\varphi_{\mathcal{F}_i, a_{i+1}}$, $\bar{t} \cup \{a_{i+1}\} \notin \mathcal{F}_i$. Therefore $\varphi_{\mathcal{F}_i, a_{i+1}}(\bar{t}) = \bar{t} \cup \{a_{i+1}\} = t$ and, if $\bar{t} \cup f \cup \{a_{i+1}\} \notin \mathcal{F}_i$ then $\varphi_{\mathcal{F}_i, a_{i+1}}(\bar{t} \cup f) = \bar{t} \cup f \cup \{a_{i+1}\} = t \cup f \in \mathcal{F}_{i+1}$. Otherwise $\bar{t} \cup f \cup \{a_{i+1}\} \in \mathcal{F}_i$, thus $t \cup f \in \mathcal{F}_i$, hence $\varphi_{\mathcal{F}_i, a_{i+1}}(t \cup f) = t \cup f \in \mathcal{F}_{i+1}$.
- $a_{i+1} \notin t$, hence necessarily $\bar{t} \cup a_{i+1} \in \mathcal{F}_i$ and $t = \bar{t}$. Thus, if $a_{i+1} \in \bar{t} \cup f$ then $\varphi_{\mathcal{F}_i, a_{i+1}}(\bar{t} \cup f) = \bar{t} \cup f = t \cup f \in \mathcal{F}_{i+1}$. Otherwise $a_{i+1} \notin \bar{t} \cup f$. Since $\bar{t} \cup a_{i+1} \in \mathcal{F}_i$, then by the inductive hypothesis $\bar{t} \cup a_{i+1} \cup f \in \mathcal{F}_i$ whence $\varphi_{\mathcal{F}_i, a_{i+1}}(\bar{t} \cup f) = \bar{t} \cup f = t \cup f \in \mathcal{F}_{i+1}$.

□

The following theorem establishes an interesting property of the associate upward-closed family $\mathcal{F} = \varphi_w(\mathcal{F})$.

Theorem 1. *For each $f \in \mathcal{F}$, φ_w is a bijection between the principal ideals*

$$\varphi_w : \mathcal{F}[f] \rightarrow \mathcal{F}[f]$$

Proof. Since φ_w is a one to one function it is sufficient to prove that $\varphi_w : \mathcal{F}[f] \rightarrow \mathcal{F}[f]$ is also surjective. Since $f \in \mathcal{F}$, then $\varphi_w(\mathcal{F}[f]) \subseteq \mathcal{F}[f]$, in particular $\mathcal{F}[f]$ is non-empty. Consider an element $\eta \in \mathcal{F}[f]$ and let $\eta^* = \varphi_w^{-1}(\eta)$. We claim that $f \subseteq \eta^*$ and so $\eta^* \in \mathcal{F}[f]$. Suppose, contrary to our claim, that $f \not\subseteq \eta^*$. Let $\eta_0 = \eta^*$ and $\eta_i = \varphi_i(\eta_0)$ for $i = 1, \dots, n$ be the trajectory of η^* . Since $f \subseteq \eta = \eta_n$, there is a minimal index $j \leq n$ such that $f \subseteq \eta_j$ and $j > 0$ ($f \not\subseteq \eta_0$). By the minimality of j , $f \not\subseteq \eta_{j-1}$. Since $f \subseteq \eta_j$, we have $a_j \notin \eta_{j-1}$ and so $a_j \in f$. By Lemma 3, since $\eta_{j-1} \in \mathcal{F}_{j-1}$, we get also $\eta_{j-1} \cup f \in \mathcal{F}_{j-1}$. Therefore, since $a_j \in f$, $f \subseteq \eta_{j-1} \cup a_j$ and so $\eta_{j-1} \cup a_j = \eta_{j-1} \cup f \in \mathcal{F}_{j-1}$, hence $f \not\subseteq \eta_j = \eta_{j-1}$, a contradiction. □

We have the following corollary:

Corollary 1. *For each $S \subseteq \mathcal{F}$, φ_w is a bijection between the principal ideals*

$$\varphi_w : \mathcal{F}[S] \rightarrow \mathcal{F}[S]$$

Moreover the inverse of $\varphi_w : \mathcal{F} \rightarrow \mathcal{F}$ is given by

$$\varphi_w^{-1}(\eta) = \bigcup_{\{f \in \mathcal{F} : f \subseteq \eta\}} f$$

Proof. Since φ_w is injective, it is sufficient to prove that it is also surjective. Thus, consider $\eta \in \mathcal{F}[S]$, then there is an $f \in S$ such that $f \subseteq \eta$, and so $\eta \in \mathcal{F}[f]$. Therefore, by Theorem 1 $\varphi_w^{-1}(\eta) \in \mathcal{F}[f] \subseteq \mathcal{F}[S]$. Let us prove the last statement, so consider an element $\eta \in \mathcal{F}$. By the previous statement $\mathcal{F} = \mathcal{F}[\min(\mathcal{F})]$. Hence the set $\{f \in \mathcal{F} : f \subseteq \eta\}$ is non-empty and so, since \mathcal{F} is union-closed:

$$\bigcup_{\{f \in \mathcal{F} : f \subseteq \eta\}} f = \eta^* \in \mathcal{F}$$

By Theorem 1 and $\eta^* \subseteq \eta$, we get $\varphi_w^{-1}(\eta) \subseteq \eta$ and $\eta^* \subseteq \varphi_w^{-1}(\eta)$. Therefore we get $\varphi_w^{-1}(\eta) \subseteq \eta^* \subseteq \varphi_w^{-1}(\eta)$, i.e. $\eta^* = \varphi_w^{-1}(\eta)$. \square

We give a lemma useful in the sequel.

Lemma 4. *The map $\psi(z) = z \cup \{a\}$ is an embedding*

$$\psi : \mathcal{F}_{\bar{a}} \hookrightarrow \varphi_w(\mathcal{F}_a)$$

Proof. Since $\psi : \mathcal{F}_{\bar{a}} \rightarrow 2^X$ is already injective, it is sufficient to prove $\psi(\mathcal{F}_{\bar{a}}) \subseteq \varphi_w(\mathcal{F}_a)$. Suppose, contrary to our claim, that there is $\eta \in \mathcal{F}_{\bar{a}}$ such that $z = \varphi_w^{-1}(\eta \cup \{a\})$ is not in \mathcal{F}_a . Since $a \notin z$ and $z \subseteq \eta \cup \{a\}$, then $z \subseteq \eta$. Let $z' = \varphi_w^{-1}(\eta)$, since $z \subseteq \eta$, then by Theorem 1 we get $z \subseteq z'$. On the other side, since $z' \subseteq \eta \subseteq \eta \cup \{a\}$ then by Theorem 1 $z' \subseteq \varphi_w^{-1}(\eta \cup \{a\}) = z$, whence $z = z'$ which implies $\eta = \eta \cup \{a\}$, a contradiction. \square

We say that $g \in \mathcal{F}$ is *fixed* by φ_w whenever $\varphi_w(g) = g$ holds. The following proposition characterized the elements of \mathcal{F} with this property.

Proposition 1. *$\varphi_w(g) = g$ if and only if $g \cup a \in \mathcal{F}$ for all $a \in X$. Moreover if $S \subseteq \mathcal{F}$ then $\mathcal{F} \cap S$ is the set of elements of S fixed by φ_w .*

Proof. Using the definition of φ_w and Lemma 3 it is straightforward to check that if $g \cup a \in \mathcal{F}$ for all $a \in X$ then $\varphi_w(g) = g$. Conversely, suppose that $\varphi_w(g) = g$ and let us prove that $g \cup a_i \in \mathcal{F}$. Since $\varphi_w(g) = g$, then if g_i is the trajectory of g in the rising process, then $g_i = g$ for all $i = 1, \dots, n$. In particular $g \cup a_{i+1} \in \mathcal{F}_i$ by definition of the rising function $\varphi_{\mathcal{F}_i, a_{i+1}}$. By Lemma 1 $a_{i+1} \in \varphi_i^{-1}(g \cup a_{i+1}) \setminus g$ and by Corollary 1 $g \subseteq \varphi_i^{-1}(g \cup a_{i+1})$, whence $g \cup a_{i+1} \subseteq \varphi_i^{-1}(g \cup a_{i+1}) \subseteq g \cup a_{i+1}$, i.e. $g \cup \{a_{i+1}\} = \varphi_i^{-1}(g \cup a_{i+1}) \in \mathcal{F}$. The proof of the last statement of the lemma is also a consequence of Corollary 1 and it is left to the reader. \square

The last proposition shows that all the upward-closed families of sets are leaved unchanged by the rising operator φ_w .

We remind that if $A \subseteq B$ are two subsets of X then the *interval* $[A, B]$ is defined by $\{D \subseteq X : A \subseteq D \subseteq B\}$. In [18, Lemma 1.3 (ii)] the author shows that if $g \neq f$ are two distinct elements of \mathcal{F} then $[g, \varphi_w(g)] \cap [f, \varphi_w(f)] = \emptyset$. This facts is independent from the order with which we rise the set, indeed we have the following proposition.

Proposition 2. *Let $f, g \in \mathcal{F}$ and $\sigma, \theta \in \mathfrak{S}_X$. Then $f \neq g$ if and only if $[f, \varphi_{w\theta}(f)] \cap [g, \varphi_{w\sigma}(g)] = \emptyset$.*

Proof. Suppose that $z \in [f, \varphi_{w\theta}(f)] \cap [g, \varphi_{w\sigma}(g)] \neq \emptyset$. By Corollary 1 and $f \subseteq z \subseteq \varphi_{w\sigma}(g)$ we get $g = \bigcup_{\{h \in \mathcal{F} : h \subseteq \varphi_{w\sigma}(g)\}} h \supseteq f$. Changing g with f we obtain the other inclusion $g \subseteq f$, whence $g = f$. The other side of the implication is trivial. \square

4 The invariant upward-closed family associated to a union-closed family

In this section we introduce an upward-closed family associated to \mathcal{F} which do not depend on a parameter like the case obtained using the rising functions in Section 3. From Theorem 1 we have that $\varphi_w(\mathcal{F})$ is an upward-closed family, moreover since the union of upward-closed families is still an upward-closed family, we can associate to \mathcal{F} the upward-closed family

$$\mathbf{U}(\mathcal{F}) = \bigcup_{\vartheta \in \mathfrak{S}_X} \varphi_{w\vartheta}(\mathcal{F}) \quad (1)$$

where $w = a_1 \dots a_n$. We call $\mathbf{U}(\mathcal{F})$ the *invariant upward-closed family* associated to \mathcal{F} . We have already noted in Section 3 that there is an action of \mathfrak{S}_X on this set given by $\beta \cdot \varphi_{w\vartheta}(g) = \varphi_{w\vartheta\beta}(g)$. So it seems natural to characterize the orbits $\mathfrak{S}_X \cdot \varphi_w(g) = \{\varphi_{w\vartheta}(g), \vartheta \in \mathfrak{S}_X\}$. Before giving this characterization we need first some definitions. The rising function φ_w depends on the parameter w , however by Corollary 1 the inverse of φ_w does not. Moreover, by the same Corollary, $\varphi_w(\mathcal{F}) \subseteq \min(\mathcal{F})^\uparrow$ and so $\mathbf{U}(\mathcal{F}) \subseteq \min(\mathcal{F})^\uparrow$. For this reason it is important to extend the map φ_w^{-1} to an operator

$$\circ^* : \min(\mathcal{F})^\uparrow \rightarrow \mathcal{F}$$

which associates to each element $z \in \min(\mathcal{F})^\uparrow$ the element

$$z^* = \bigcup_{\{h \in \mathcal{F}, h \subseteq z\}} h$$

Using the fact that \mathcal{F} is \cup -closed and the domain is $\min(\mathcal{F})^\uparrow$, it is immediate to see that this operator is well defined. Moreover \circ^* preserves inclusion, i.e. if $z \subseteq y$ then $z^* \subseteq y^*$ and it is clearly surjective, thus we can define the *fiber* of each $g \in \mathcal{F}$ as

$$\text{Fib}(g) = \{h \in \min(\mathcal{F})^\uparrow : h^* = g\}.$$

The following proposition characterizes the union-closed families in term of the operator \circ^* .

Proposition 3. *Let \mathcal{H} be a family of subsets of X and consider the operator*

$$\circ^* : \min(\mathcal{H})^\uparrow \rightarrow 2^X$$

defined by sending each $z \in \min(\mathcal{H})^\uparrow$ into $z^ = \bigcup_{\{h \in \mathcal{H}, h \subseteq z\}} h$. Then \mathcal{H} is a \cup -closed family if and only if the image of \circ^* is contained in \mathcal{H} .*

Proof. As we have already noticed before if \mathcal{H} is \cup -closed then \circ^* is well defined map $\circ^* : \min(\mathcal{H})^\uparrow \rightarrow \mathcal{H}$. Conversely, let $g, h \in \mathcal{H}$ and let $\mathcal{H}' \subseteq \mathcal{H}$ be the image of \mathcal{H} by means of \circ^* . The element $g \cup h \in \min(\mathcal{H})^\uparrow$ and so $g \cup h \in \text{Fib}(t)$ for some $t \in \mathcal{H}'$. Since $\text{Fib}(t)$ is formed by the elements z such that $z^* = t$ and $t \in \mathcal{H}'$ we have that $t \subseteq z$ for all $z \in \text{Fib}(t)$, in particular $t \subseteq g \cup h$. On the other hand $g, h \subseteq g \cup h$ and so $g, h \subseteq (g \cup h)^* = t$, whence $g \cup h \subseteq t$ and so $g \cup h = t \in \mathcal{H}' \subseteq \mathcal{H}$. \square

Given a word $u = w\theta = a_{i_1} \dots a_{i_n}$ for some $\theta \in \mathfrak{S}_X$ and a subset $\gamma \subseteq X = \{a_1, \dots, a_n\}$ we say that γ is *contained in a prefix* of u (or u has a prefix containing γ) whenever either γ is empty or there is a prefix $u' = a_{i_1} \dots a_{i_l}$ of u for some l with $n \geq l \geq 1$ with $\gamma = \{a_{i_1}, \dots, a_{i_l}\}$.

Lemma 5. *Let \mathcal{F} be a \cup -closed family of sets of $X = \{a_1, \dots, a_n\}$, let $g \in \mathcal{F}$ and $\eta \in \max(\text{Fib}(g))$. Then for any word $u = a_{i_1} \dots a_{i_n}$ having a prefix containing $\eta \setminus \eta^*$ we have $\varphi_u(g) = \eta$.*

Proof. Suppose $\eta \setminus \eta^* \neq \emptyset$ (the empty case can be treated analogously) and let $u = a_{i_1} \dots a_{i_n}$ be a word with the property of the statement and so there is some l with $n \geq l \geq 1$ such that $\eta \setminus \eta^* = \{a_{i_1}, \dots, a_{i_l}\}$. Let $\eta_0 = g$ and $\eta_j = \varphi_j(g)$ for $j = 1, \dots, n$ be the trajectory of g trough the iterated application of the rising functions with respect to u and let \mathcal{F}_j be the associated sections. Suppose that there is an integer s with $0 \leq s < l$ such that $\eta_s \cup a_{i_{s+1}} \in \mathcal{F}_s$ and let us suppose without loss of generality that such s is minimum between the integers with this property. Since $\eta_s \cup a_{i_{s+1}} \in \mathcal{F}_s$ there is an element $f \in \mathcal{F}$ with $f \neq g$ such that $\eta_s \cup a_{i_{s+1}} = \varphi_s(f)$. Thus,

since $g \subseteq \eta_s$ we get $g \subseteq \eta_s \cup a_{i_{s+1}} = \varphi_s(f) \subseteq \varphi_w(f)$ and so by Corollary 1 we have $g \subsetneq f$. Since $a_{i_1} \dots a_{i_l}$ is a prefix of u and $\{a_{i_1}, \dots, a_{i_l}\} = \eta \setminus \eta^*$, $g = \eta_0 \subseteq \eta$ then $\eta_s \subseteq \eta$, moreover since $s < l$ then $a_{i_{s+1}} \in \eta$, hence $\eta_s \cup a_{i_{s+1}} \subseteq \eta$. Therefore we have the contradiction:

$$g = \eta^* \supseteq (\eta_s \cup a_{i_{s+1}})^* = f \supsetneq g$$

since by Corollary 1 $f = \varphi_u(f)^* \supseteq (\eta_s \cup a_{i_{s+1}})^* \supseteq f$. Hence we can suppose that for all $0 \leq s < l$ we have $\eta_s \cup a_{i_{s+1}} \notin \mathcal{F}_s$ and so we have $\eta_l = \eta$. Thus $\eta \subseteq \varphi_u(g)$. Let us prove that actually $\eta = \varphi_u(g)$. Suppose on the contrary that $\eta \subsetneq \varphi_u(g)$, since by Corollary 1 $g = (\varphi_u(g))^*$ then $\varphi_u(g) \in \text{Fib}(g)$, however $\eta \subsetneq \varphi_{w'}(g)$ contradicts the maximality of η , hence $\eta = \varphi_u(g)$. \square

The following theorem characterizes the orbits of $\mathbf{U}(\mathcal{F})$.

Theorem 2. *Let \mathcal{F} be a \cup -closed family of sets of $X = \{a_1, a_2, \dots, a_n\}$ and let $w = a_1 a_2 \dots a_n$. Let $g \in \mathcal{F}$ then:*

$$\mathfrak{S}_X \cdot \varphi_w(g) = \max(\text{Fib}(g))$$

Proof. The inclusion $\max(\text{Fib}(g)) \subseteq \mathfrak{S}_X \cdot \varphi_w(g)$ is a consequence of Lemma 5. On the other hand, let $\varphi_{w'}(g) \in \mathfrak{S}_X \cdot \varphi_w(g)$ for some $w' = w\theta$, $\theta \in \mathfrak{S}_X$. By Corollary 1 $(\varphi_{w'}(g))^* = g$, thus we have $\varphi_{w'}(g) \in \text{Fib}(g)$. Suppose, contrary to the statement of the lemma, that $\varphi_{w'}(g)$ is not maximal in $\text{Fib}(g)$ and so let $\eta' \in \text{Fib}(g)$ such that $\varphi_{w'}(g) \subsetneq \eta'$. Since $\varphi_{w'}(\mathcal{F})$ is an upward-closed set and $\varphi_{w'}(g) \in \varphi_{w'}(\mathcal{F})$ with $\varphi_{w'}(g) \subsetneq \eta'$, then we get $\eta' \in \varphi_{w'}(\mathcal{F})$. However, by Corollary 1 we have the contradiction $g \subsetneq (\eta')^* = g$. Hence $\varphi_{w'}(g) \in \max(\text{Fib}(g))$ and so $\mathfrak{S}_X \cdot \varphi_w(g) \subseteq \max(\text{Fib}(g))$. \square

Note that Theorem 2, together with the fact that $\text{Fib}(g) \cap \text{Fib}(f) = \emptyset$ iff $g \neq f$, implies Proposition 2, in particular we have

$$\text{Fib}(g) = \bigcup_{\vartheta \in \mathfrak{S}_X} [g, \varphi_{w\vartheta}(g)]$$

Using the invariant upward-closed family $\mathbf{U}(\mathcal{F})$ we can give tight upper and lower bounds to $|\mathcal{F}|$ depending on $rk(\mathcal{F}) = \min\{|\eta| : \eta \in \min(\mathbf{U}(\mathcal{F}))\}$. We have the following proposition:

Proposition 4.

$$2^{n-rk(\mathcal{F})} \leq |\mathcal{F}| \leq \sum_{i \geq rk(\mathcal{F})} \binom{n}{i}$$

and these bounds are tight.

Proof. Let $z \in \min(\mathbf{U}(\mathcal{F}))$ with $|z| = rk(\mathcal{F})$, then by Theorem 2 $z = \varphi_{w\theta}(g)$ for some $\theta \in \mathfrak{S}_X$, thus $z^\uparrow \subseteq \varphi_{w\theta}(\mathcal{F})$. Thus $|\mathcal{F}| = |\varphi_{w\theta}(\mathcal{F})| \geq |z^\uparrow| = 2^{n-rk(\mathcal{F})}$. This bound is attained considering the \cup -closed family $\{\bar{z}\}^\uparrow$. By Proposition 1 $\varphi_{w\theta}(\mathcal{F}) = \{\bar{z}\}^\uparrow$ for all $\theta \in \mathfrak{S}_X$, thus $\mathbf{U}(\mathcal{F}) = \{\bar{z}\}^\uparrow$ and so $rk(\mathcal{F}) = |\bar{z}|$. The upper bound is obtained in a similar way and its proof is left to the reader. \square

Let $x \in \mathbf{U}(\mathcal{F})$, \mathfrak{S}_x denotes the stabilizer subgroup of x and as usual by $\mathbf{U}(\mathcal{F})^\vartheta$ the set of elements of $\mathbf{U}(\mathcal{F})$ fixed by an element $\vartheta \in \mathfrak{S}_X$. As a consequence of Theorem 2 and Burnside's Lemma we have the following corollary:

Corollary 2.

$$|\mathcal{F}| = \frac{1}{n!} \sum_{x \in \mathbf{U}(\mathcal{F})} |\mathfrak{S}_x|$$

In particular we have the following inequality:

$$\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \sum_{x \in \max(Fib(f))} \frac{1}{\binom{n}{|x \setminus x^*|}} \leq 1$$

Proof. Using Burnside's Lemma

$$|\mathbf{U}(\mathcal{F})/\mathfrak{S}_X| = \frac{1}{|\mathfrak{S}_X|} \sum_{\vartheta \in \mathfrak{S}_X} |\mathbf{U}(\mathcal{F})^\vartheta| = \frac{1}{n!} \sum_{x \in \mathbf{U}(\mathcal{F})} |\mathfrak{S}_x|$$

by Theorem 2 the set of orbits $\mathbf{U}(\mathcal{F})/\mathfrak{S}_X$ is in one to one correspondence with \mathcal{F} , thus $|\mathbf{U}(\mathcal{F})/\mathfrak{S}_X| = |\mathcal{F}|$ and so the equality of the corollary is proved. To prove the inequality we give a lower bound to $|\mathfrak{S}_x|$ for $x \in \max(Fib(f))$. By Lemma 5 we have that for any word $u = a_{i_1} \dots a_{i_n}$ having a prefix containing $x \setminus x^*$, $\varphi_u(g) = x$. There are $|x \setminus x^*|!(n - |x \setminus x^*|)!$ such words and so $|\mathfrak{S}_x| \geq |x \setminus x^*|!(n - |x \setminus x^*|)!$. Therefore by Theorem 2 we have

$$\begin{aligned} 1 &= \frac{1}{|\mathcal{F}|n!} \sum_{x \in \mathbf{U}(\mathcal{F})} |\mathfrak{S}_x| \geq \frac{1}{|\mathcal{F}|} \sum_{x \in \mathbf{U}(\mathcal{F})} \frac{(|x \setminus x^*|)!(n - |x \setminus x^*|)!}{n!} \\ &= \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \sum_{x \in \max(Fib(f))} \frac{1}{\binom{n}{|x \setminus x^*|}} \end{aligned}$$

\square

The following lemma characterizes the elements not containing an $a \in X$ for which in the rising process, for some order of rising, the elements will also not contain a .

Lemma 6. *Let $g \in \mathcal{F}_{\bar{a}}$ and $\eta \in \max(\text{Fib}(g))$. Then $a \notin \eta$ if and only if there is $h \in \mathcal{F}_a$ such that $h \subseteq \eta \cup \{a\}$ and $g \leq h$.*

Proof. Suppose that $a \notin \eta$ and let $h' = (\eta \cup \{a\})^*$. Since $\eta \in \text{Fib}(g)$, then $g \subseteq \eta$ and so $g \subseteq h'$. We claim $(h' \setminus \{a\})^* = g$. By Lemma 4 and Theorem 2 we get $h' \in \mathcal{F}_a$, and by definition of the operator \circ^* , $h' \subseteq \eta \cup \{a\}$. Since $g \subseteq h'$ and $g \in \mathcal{F}_{\bar{a}}$, then $g \subseteq (h' \setminus \{a\})^* \subseteq \eta^* = g$ and so the claim $(h' \setminus \{a\})^* = g$. Reasoning by contradiction suppose that there is a $g' \in \mathcal{F}_{\bar{a}}$ such that $g \leq g' \subseteq h'$. Thus $g' \subseteq (h' \setminus \{a\})$ and so we get the contradiction $g \leq g' \subseteq (h' \setminus \{a\})^* = g$, whence there is an $h \in \mathcal{F}_a$ such that $g \leq h \subseteq h'$. On the other side, suppose, contrary to the statement of the lemma, that $a \in \eta$. Thus $h \subseteq \eta$, hence we have $h \subseteq \eta^* = g$. However $h \in \mathcal{F}_a$ and $g \in \mathcal{F}_{\bar{a}}$, a contradiction. \square

The following lemma characterizes the elements of $\mathcal{F}_{\bar{a}}$ that have at least one maximal element in their fiber that do not contain the element a .

Lemma 7. *Let $a \in X$ and let $g \in \mathcal{F}_{\bar{a}}$. Then there is an $\eta \in \max(\text{Fib}(g))$ with $a \notin \eta$ if and only if there is an $h \in \mathcal{F}_a$ such that $g \leq h$.*

Proof. Suppose that there is an $\eta \in \max(\text{Fib}(g))$ with $a \notin \eta$. By Lemma 6 there is an $h \in \mathcal{F}_a$ such that $g \leq h$. We prove the other side of the equivalence using an argument similar to the one in Lemma 6. Indeed consider the permutation $(a_{i_1}, \dots, a_{i_n})$ of X with $g = \{a_{i_1}, \dots, a_{i_k}\}$, $h = \{a_{i_1}, \dots, a_{i_l}\}$, $a_{i_l} = a$ for some $n \geq l \geq k$. Consider the word $w' = a_{i_1}, \dots, a_{i_n}$, put $\eta_0 = g$ and $\eta_j = \varphi_j(\eta_0)$ for $j = 1, \dots, n$ be the trajectory of g trough the iterated application of the rising functions with respect to w' and let \mathcal{F}_j be the associated sections. We claim that $\varphi_{l-1}(g) = g \cup \{a_{i_{k+1}}, \dots, a_{i_{l-1}}\}$. Clearly $\eta_k = g$ and suppose, contrary to our claim, that there is an integer s with $k \leq s < l-1$ such that $\eta_s \cup a_{i_{s+1}} \in \mathcal{F}_s$ and let us suppose that s is the minimum between the integers with this property. Since $\eta_s \cup a_{i_{s+1}} \in \mathcal{F}_s$ there is an element $g' \in \mathcal{F}$ with $g' \neq g$ such that $\eta_s \cup a_{i_{s+1}} = \varphi_s(g')$. Thus, since $g \subseteq \eta_s$ we get $g \subseteq \eta_s \cup a_{i_{s+1}} = \varphi_s(g) \subseteq \varphi_{w'}(g')$ and so by Corollary 1 we have $g \subsetneq g'$. Since $a_{i_1} \dots a_{i_l}$ is a prefix of w' , $h = \{a_{i_1}, \dots, a_{i_l}\}$, $\eta_0 \subseteq h$, $s < l-1$ and $a_{i_l} = a$ then $\eta_s \cup a_{i_{s+1}} \subseteq h \setminus \{a\}$. Since $g \leq h$ and $g \in \mathcal{F}_{\bar{a}}$ it is straightforward to check that $g = (h \setminus \{a\})^*$ and so we have the contradiction:

$$g = (h \setminus \{a\})^* \supseteq (\eta_s \cup a_{i_{s+1}})^* = g' \supsetneq g$$

since by Corollary 1 we have $g' = \varphi_{w'}(g')^* \supseteq (\eta_s \cup a_{i_{s+1}})^* \supseteq g'$. Therefore $\eta_s \cup a_{i_{s+1}} \notin \mathcal{F}_s$ for all $k \leq s < l-1$ and so $\varphi_{l-1}(g) = g \cup \{a_{i_{k+1}}, \dots, a_{i_{l-1}}\}$. Since $h = \varphi_{l-1}(h) \in \mathcal{F}_{l-1}$ and $\varphi_{l-1}(g) \cup \{a_{i_l}\} = h$ we have $\varphi_{l-1}(g) \cup \{a_{i_l}\} \in$

\mathcal{F}_{l-1} hence $\varphi_l(g) = \varphi_{l-1}(g) = h \setminus a$ ($a = a_{i_l}$) and so $a \notin \eta_m$ for all $m \geq l$. In particular $a \notin \varphi_{w'}(g)$, whence by Theorem 2 $\varphi_{w'}(g) \in \max(\text{Fib}(g))$ is the element η satisfying the condition of the lemma. \square

In view of Lemma 7 we say that $g \in \mathcal{F}_{\bar{a}}$ is *covered in a* if there is an $h \in \mathcal{F}_a$ such that $g < h$. In this case we say that h covers g in a . The following proposition gives an equivalent formulation of this definition.

Proposition 5. *$g \in \mathcal{F}_{\bar{a}}$ is covered in a iff there is an $h \in \mathcal{F}_a$ such that $(h \setminus \{a\})^* = g$.*

Proof. Suppose that $h \in \mathcal{F}_a$ such that $g < h$, then it is straightforward to see that $(h \setminus \{a\})^* = g$. Conversely suppose that there is an $h \in \mathcal{F}_a$ such that $(h \setminus \{a\})^* = g$. Arguing by contradiction suppose that g is not covered in a and so for any $t \in \mathcal{F}_a$ there is a $g' \in \mathcal{F}_{\bar{a}}$ such that $g \subsetneq g' \subsetneq t$. In particular this occurs for h , hence there is a $g' \in \mathcal{F}_{\bar{a}}$ with $g \subsetneq g' \subsetneq h$. Thus we have the contradiction $g' \subseteq (h \setminus \{a\})^* = g \subsetneq g'$. \square

From this proposition we have that the set

$$\text{Cov}_a(g) = \{h \in \mathcal{F}_a : (h \setminus \{a\})^* = g\}$$

is non-empty iff g is covered in a .

5 Some results around Frankl's conjecture

The connection between upward-closed families and \cup -closed families that we have established in the previous two sections can be useful to try to tackle Frankl's conjecture. The aim of this section is to introduce some subsets which are related to this conjecture. In particular in the first part we fix a word w and we introduce these sets using the rising function φ_w , in the second part we draw some consequences of this approach giving some lower bounds on the quantity $\frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} |f|$, for any $S \subseteq \mathcal{F}$, and in the last part we consider the invariant case.

5.1 Some useful subsets

Definition 1. Let \mathcal{H} be a family of sets of $X = \{a_1, \dots, a_n\}$ and let $a \in X$. We denote by $S(\mathcal{H}, a)$ the set of all the elements $z \in \mathcal{H}$ such that $z \cup \{a\} \notin \mathcal{H}$. Dually we put $P(\mathcal{H}, a)$ as the set of all the elements $z \in \mathcal{H}$ such that $z \setminus \{a\} \notin \mathcal{H}$.

Note that $P(\mathcal{H}, a)$ is non-empty since $\min\{\mathcal{H}\}_a \subseteq P(\mathcal{H}, a)$. We have the following proposition.

Proposition 6. *Let \mathcal{H} be a family of sets of X , then for any $a \in X$:*

$$|\mathcal{H}_a| - |\mathcal{H}_a^c| = |P(\mathcal{H}, a)| - |S(\mathcal{H}, a)|$$

Moreover if \mathcal{H} is \cup -closed, then Frankl's conjecture holds for \mathcal{H} if and only if there is some $a \in X$ such that

$$|P(\mathcal{H}, a)| \geq |S(\mathcal{H}, a)|$$

Proof. It is straightforward to check that the function ψ from the set $\{f \in \mathcal{H}_a : f \setminus \{a\} \in \mathcal{H}\}$ onto the set $\{f \in \mathcal{H}_a^c : f \cup \{a\} \in \mathcal{H}\}$ defined by $\psi(z) = z \setminus \{a\}$ is a bijection. Furthermore $\{f \in \mathcal{H}_a : f \setminus \{a\} \in \mathcal{H}\}$ is in bijection with the set $\{f \in \mathcal{H}_a^c : f \cup \{a\} \in \mathcal{H}^c\}$ and so $|\{f \in \mathcal{H}_a^c : f \cup \{a\} \in \mathcal{H}^c\}| = |\{f \in \mathcal{H}_a : f \cup \{a\} \in \mathcal{H}\}|$, whence

$$\begin{aligned} |\mathcal{H} \setminus \{f \in \mathcal{H}_a : f \cup \{a\} \in \mathcal{H}\}| &= |\mathcal{H}| - |\{f \in \mathcal{H}_a : f \cup \{a\} \in \mathcal{H}\}| = \\ &= |\mathcal{H}^c| - |\{f \in \mathcal{H}_a^c : f \cup \{a\} \in \mathcal{H}^c\}| = |\mathcal{H}^c \setminus \{f \in \mathcal{H}_a^c : f \cup \{a\} \in \mathcal{H}^c\}| \end{aligned}$$

Hence from $|\mathcal{H} \setminus \{f \in \mathcal{H}_a : f \cup \{a\} \in \mathcal{H}\}| = |\mathcal{H}^c \setminus \{f \in \mathcal{H}_a^c : f \cup \{a\} \in \mathcal{H}^c\}|$ we get the equality

$$|\mathcal{H}_a| + |\{f \in \mathcal{H}_a^c : f \cup \{a\} \notin \mathcal{H}\}| = |\mathcal{H}_a^c| + |\{f \in \mathcal{H}_a^c : f \cup \{a\} \notin \mathcal{H}^c\}|$$

and so the statement follows from $|\mathcal{H}_a^c| = |\mathcal{H}_a|$, $|\{f \in \mathcal{H}_a^c : f \cup \{a\} \notin \mathcal{H}^c\}| = |\{f \in \mathcal{H}_a : f \setminus \{a\} \notin \mathcal{H}\}| = |P(\mathcal{H}, a)|$ and $\{f \in \mathcal{H}_a^c : f \cup \{a\} \notin \mathcal{H}^c\} = S(\mathcal{H}, a)$. The last claim of the proposition is a consequence of $2|\mathcal{H}_a| - |\mathcal{H}| = |\mathcal{H}_a| - |\mathcal{H}_a^c|$. \square

Therefore the study of the sets $S(\mathcal{H}, a)$ and $P(\mathcal{H}, a)$ seems important in a possible proof of the Frankl's conjecture. Let us fix a \cup -closed family \mathcal{F} on X , let $\mathcal{F} = \varphi_w(\mathcal{F})$ be the associated upward-closed family for some fixed word w . We introduce now two analogous sets which are important to give a lower bound to the quantity $\frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} |f|$, for any $S \subseteq \mathcal{F}$ and which are somehow related to $S(\mathcal{H}, a)$ and $P(\mathcal{H}, a)$.

Definition 2. *Let $a \in X$, the set $\sigma_w(\mathcal{F}, a) = \{\eta \in \varphi_w(\mathcal{F}) : a \in \eta \setminus \varphi_w^{-1}(\eta)\}$ is called the set of spurious elements of \mathcal{F} with respect to a . The set $\pi_w(\mathcal{F}, a) = \{\eta \in \varphi_w(\mathcal{F}_a) : \eta \setminus \{a\} \notin \varphi_w(\mathcal{F})\}$ is called the set of pure elements of \mathcal{F} with respect to a .*

Let $\eta \in \mathcal{F}$, the set of pure elements of η , denoted by $\pi_w(\mathcal{F}, \eta)$, is the set $\{a \in X : \eta \in \pi_w(a)\}$ and analogously the set of spurious elements of η is the set $\sigma_w(\mathcal{F}, \eta) = \{a \in X : \eta \in \sigma_w(a)\}$.

When the \cup -closed set \mathcal{F} is clear from the context, we drop \mathcal{F} from $\sigma_w(\mathcal{F}, a), \sigma_w(\mathcal{F}, \eta), \pi_w(\mathcal{F}, a), \pi_w(\mathcal{F}, \eta)$ and we use instead $\sigma_w(a), \sigma_w(\eta), \pi_w(a), \pi_w(\eta)$. We have the following lemma.

Lemma 8. *The two sets $\varphi_w(\mathcal{F}_a), \sigma_w(a)$ form a partition of \mathcal{F}_a . In turn $\varphi_w(\mathcal{F}_a)$ is partitioned by $\pi_w(a), \psi(\mathcal{F}_{\bar{a}})$ where $\psi(z) = z \cup \{a\}$. Moreover $\sigma_w(a) \cup \pi_w(a) = \{z \in \mathcal{F} : z \setminus \{a\} \notin \mathcal{F}\}$ and*

$$|\mathcal{F}_a| = |\mathcal{F}_{\bar{a}}| + |\pi_w(a)| + |\sigma_w(a)|.$$

Proof. Since $\sigma_w(a) \subseteq \mathcal{F}_a$ and $\varphi_w(\mathcal{F}_a) \subseteq \mathcal{F}_a$, then $\mathcal{F}_a \setminus \varphi_w(\mathcal{F}_a)$ is formed by elements $z \in \mathcal{F}_a$ for which a is a spurious element of z , i.e. $\mathcal{F}_a \setminus \varphi_w(\mathcal{F}_a) = \sigma_w(a)$. By Lemma 4 $\psi(\mathcal{F}_{\bar{a}}) \subseteq \varphi_w(\mathcal{F}_a)$ and if $z \in \varphi_w(\mathcal{F}_a) \setminus \psi(\mathcal{F}_{\bar{a}})$ then $z \setminus \{a\} \notin \mathcal{F}$, otherwise $z = \psi(z \setminus \{a\})$. Therefore $\pi_w(a) = \varphi_w(\mathcal{F}_a) \setminus \psi(\mathcal{F}_{\bar{a}})$. By the previous statements it is also evident that:

$$\sigma_w(a) \cup \pi_w(a) = \mathcal{F}_a \setminus \psi(\mathcal{F}_{\bar{a}}) = \{z \in \mathcal{F} : z \setminus \{a\} \notin \mathcal{F}\}$$

Since \mathcal{F}_a is partitioned into the two sets and $\sigma_w(a) \varphi_w(\mathcal{F}_a)$ which in turn is partition by the two sets $\pi_w(a), \psi(\mathcal{F}_{\bar{a}})$, and ψ is an injective map we have:

$$|\mathcal{F}_a| = |\psi(\mathcal{F}_{\bar{a}})| + |\sigma_w(a)| + |\pi_w(a)| = |\mathcal{F}_{\bar{a}}| + |\sigma_w(a)| + |\pi_w(a)|$$

and this completes the proof of the lemma. \square

The following proposition gives an alternative formulation of Frankl's conjecture which is the analogous of Proposition 6.

Proposition 7. *For any $a \in X$*

$$|\pi_w(a)| - |\sigma_w(a)| = |P(\mathcal{F}, a)| - |S(\mathcal{F}, a)|$$

and so Frankl's conjecture holds for \mathcal{F} if and only if $|\pi_w(a)| \geq |\sigma_w(a)|$ for some $a \in X$. Moreover $|\pi_w(a)| \leq |P(\mathcal{F}, a)|, |\sigma_w(a)| \leq |S(\mathcal{F}, a)|$.

Proof. It is not difficult to check that $|\mathcal{F}_{\bar{a}}| - |\sigma_w(a)| = |\mathcal{F}_{\bar{a}}|$ and by Lemma 8 we have $|\mathcal{F}_a| = |\mathcal{F}_{\bar{a}}| + |\sigma_w(a)|$. Thus by the same Lemma 8 we get

$$|\mathcal{F}_a| = |\mathcal{F}_{\bar{a}}| + |\pi_w(a)| - |\sigma_w(a)|$$

and so, by Proposition 6 we get the statement $|P(\mathcal{F}, a)| - |S(\mathcal{F}, a)| = |\mathcal{F}_a| - |\mathcal{F}_{\bar{a}}| = |\pi_w(a)| - |\sigma_w(a)|$.

Let us prove the last statement showing that $\sigma_w(a) \subseteq \varphi_w(S(\mathcal{F}, a))$. Let $\eta \in \sigma_w(a)$. Reasoning by contradiction, suppose that $z = \varphi_w^{-1}(\eta) \notin S(\mathcal{F}, a)$

and so $z \cup \{a\} \in \mathcal{F}$. Since $z \cup \{a\} \subseteq \eta$, by Corollary 1 we get $\eta \in \mathcal{F}[z \cup \{a\}] \simeq \mathcal{F}[z \cup \{a\}]$ and so $a \in z \cup \{a\} \subseteq \varphi_w^{-1}(\eta) = z$ which contradicts $\eta \in \sigma_w(a)$. The statement $|\pi_w(a)| \leq |P(\mathcal{F}, a)|$ is a consequence of $|\pi_w(a)| - |\sigma_w(a)| = |P(\mathcal{F}, a)| - |S(\mathcal{F}, a)|$ and $|\sigma_w(a)| \leq |S(\mathcal{F}, a)|$. \square

In view of Proposition 7 it is interesting to give a lower bound to the set $|\pi_w(a)|$. The following proposition gives a partial answer, we recall that \circ^* is the operator introduced in Section 4.

Proposition 8. *For any $a \in X$ we have:*

$$\{\varphi_w(g) : g \in \mathcal{F}_a, (g \setminus \{a\})^* = \emptyset\} \subseteq \pi_w(a)$$

Proof. Let $g \in \mathcal{F}_a, (g \setminus \{a\})^* = \emptyset$ and suppose, contrary to the statement, that $\varphi_w(g) \setminus \{a\} \in \mathcal{F}$. By Corollary 1 we have

$$\varphi_w^{-1}(\varphi_w(g) \setminus \{a\}) = \bigcup_{f \subseteq \varphi_w(g) \setminus \{a\}} f \subseteq \bigcup_{f \subseteq g \setminus \{a\}} f = (g \setminus \{a\})^*$$

whence $(g \setminus \{a\})^* \neq \emptyset$, a contradiction. \square

We remark that the set $\{g \in \mathcal{F}_a : (g \setminus \{a\})^* = \emptyset\}$ is non-empty since it contains $\min(\mathcal{F})_a$.

The subsets $\pi_w(\eta), \sigma_w(\eta)$ introduced in Definition 2 are the “local” version of $\pi_w(a), \sigma_w(a)$ in the following sense:

$$\sum_{a \in X} |\pi_w(a)| = \sum_{\eta \in \mathcal{F}} |\pi_w(\eta)|, \quad \sum_{a \in X} |\sigma_w(a)| = \sum_{\eta \in \mathcal{F}} |\sigma_w(\eta)|$$

We also note that by Lemma 8 $\pi_w(\eta), \sigma_w(\eta)$ are two disjoint subsets of η and in particular by the definition we get $\sigma_w(\eta) = \eta \setminus \varphi_w^{-1}(\eta)$. The interest in introducing such subsets is given by the following characterization:

Proposition 9. *For any $\eta \in \mathcal{F}$ we have:*

$$\sigma_w(\eta) = \bigcap_{\xi \subseteq \eta} \sigma_w(\xi), \quad \pi_w(\eta) = \bigcap_{\xi \subseteq \eta} \xi \cap \varphi_w^{-1}(\eta)$$

Proof. By Lemma 8, $\sigma_w(a) \cup \pi_w(a) = \{z \in \mathcal{F} : z \setminus \{a\} \notin \mathcal{F}\}$, thus it is straightforward to check

$$\pi_w(\eta) \cup \sigma_w(\eta) = \{a \in X : \eta \setminus \{a\} \notin \mathcal{F}\} = \bigcap_{\xi \subseteq \eta} \xi$$

Since $\pi_w(\eta) \subseteq \varphi_w^{-1}(\eta)$ and $\sigma_w(\eta) = \eta \setminus \varphi_w^{-1}(\eta)$ then

$$\pi_w(\eta) = \bigcap_{\xi \subseteq \eta} \xi \cap \varphi_w^{-1}(\eta), \quad \sigma_w(\eta) = \bigcap_{\xi \subseteq \eta} \xi \cap \sigma_w(\eta)$$

We claim that if $\xi \subseteq \eta$ then $\sigma_w(\eta) \subseteq \sigma_w(\xi)$ from which it follows $\sigma_w(\eta) = \bigcap_{\xi \subseteq \eta} \sigma_w(\xi)$. Indeed by Lemma 4 for all $b \in \eta \setminus \xi$, $b \in \varphi_w^{-1}(\xi \cup \{b\})$. Thus, since $\xi \cup \{b\} \subseteq \eta$, by Theorem 1, $b \in \varphi_w^{-1}(\eta)$, whence $\eta \setminus \xi \subseteq \varphi_w^{-1}(\eta)$. By Theorem 1 we also get $\varphi_w^{-1}(\xi) \subseteq \varphi_w^{-1}(\eta)$, thus $\eta \setminus \xi \cup \varphi_w^{-1}(\xi) \subseteq \varphi_w^{-1}(\eta)$ from which we obtain $\sigma_w(\eta) \subseteq \sigma_w(\xi)$. \square

If $f \subsetneq g$ for some $f, g \in \mathcal{F}$, then in general $\sigma_w(\varphi_w(f)) \subsetneq \sigma_w(\varphi_w(g))$ do not hold. However if we keep the freedom to choose the order of the rising we can have this property. With the notation of Section 4 we have the following:

Lemma 9. *Let $f, g \in \mathcal{F}$ with $f \subseteq g$ and let $\eta \in \max(\text{Fib}(g))$, then there is a word $w' = a_{i_1} \dots a_{i_n}$ such that $\eta = \varphi_{w'}(g)$ and*

$$\sigma_{w'}(\varphi_{w'}(g)) \subseteq \sigma_{w'}(\varphi_{w'}(f))$$

Proof. Let us prove that $\eta' = (\eta \setminus g) \cup f \in \text{Fib}(f)$. It is obvious that $f \subseteq \eta'$, a let us assume, contrary to our claim, that there is $h \in \mathcal{F}$ such that $h \subseteq \eta'$ with $f \subsetneq h$. Thus $(h \setminus f) \cap (\eta \setminus g) \neq \emptyset$. Since $\eta' \subseteq \eta$, then $h \subseteq \eta$ and so $h \subseteq \eta^* = g$. In particular we have $(h \setminus f) \subseteq g$ which contradicts $(h \setminus f) \cap (\eta \setminus g) \neq \emptyset$. Therefore $\eta' \in \text{Fib}(f)$, and let $\nu \in \max(\text{Fib}(f))$ such that $\eta' \subseteq \nu$. Then we have

$$\eta \setminus \eta^* = \eta \setminus g = \eta' \setminus f \subseteq \nu \setminus f = \nu \setminus \nu^* \quad (2)$$

If we prove that there is a word w' such that $\eta = \varphi_{w'}(g)$, $\nu = \varphi_{w'}(f)$ then we have proved the statement of the lemma since (2) holds and $\sigma_{w'}(\eta) = \eta \setminus \eta^*$, $\sigma_{w'}(\nu) = \nu \setminus \nu^*$. Since $\eta \setminus \eta^* \subseteq \nu \setminus \nu^*$, then we can find a word w' such that both $\eta \setminus \eta^*$ and $\nu \setminus \nu^*$ are contained in a prefix of w' , hence by Lemma 5, we have $\eta = \varphi_{w'}(g)$, $\nu = \varphi_{w'}(f)$. \square

5.2 The average length

The *average of the length* of the elements of \mathcal{F} , simply the *average* of the family \mathcal{F} , is the integer $\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} |f|$, this number is important because the following well known equality holds

$$\sum_{a \in X} \frac{|\mathcal{F}_a|}{|\mathcal{F}|} = \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} |f|$$

For instance the averaged Frankl's property $\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} |f| \geq \frac{n}{2}$ implies that Frankl's conjecture is true for \mathcal{F} . Unfortunately the converse is not true, indeed it is a well know fact that many union-closed families fail to satisfy the averaged Frankl's property (see [7, 8]). However the average of \mathcal{F} is still an interesting parameter at least because any lower bound on it gives rise to a lower bound of $\max_{a \in X} \{|\mathcal{F}_a|/|\mathcal{F}|\}$. In [18] Reimer shows that

$$\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} |f| \geq \frac{1}{2} \log_2(|\mathcal{F}|)$$

and in [12] the bound is improved in the case of a separating family. What we consider here is the localized version of the average of \mathcal{F} , given a subfamily $S \subseteq \mathcal{F}$, the *average of \mathcal{F} localized on S* is defined by

$$\frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} |f|$$

and gives the average of the length of the elements contained in the principal ideal of \mathcal{F} generated by S . Our aim is to provide lower bounds to such quantity. Note that we can assume without loss of generality that S is an antichain. We fix the notation and for the rest of the section \mathcal{F} denotes a \cup -closed family of sets of $X = \{a_1, \dots, a_n\}$, $S \subseteq \mathcal{F}$ is an antichain, and $\mathcal{F} = \varphi_w(\mathcal{F})$ is the upward-closed family associate to \mathcal{F} with respect to the word $w = a_1 a_2 \dots a_n$.

Proposition 10. *The following bound holds:*

$$\sum_{f \in \mathcal{F}[S]} |f| \geq \frac{n}{2} |\mathcal{F}[S]| + \frac{1}{2} \sum_{a \in X} |\pi_w(a) \cap S^\uparrow| - |\sigma_w(a) \cap S^\uparrow|.$$

with equality if $S = \min(\mathcal{F})$.

Proof. By Lemma 8 there is a partition $\mathcal{F}_a = \psi(\mathcal{F}_{\bar{a}}) \cup \pi_w(a) \cup \sigma_w(a)$, hence:

$$\mathcal{F}_a[S] = (\psi(\mathcal{F}_{\bar{a}}) \cap S^\uparrow) \cup (\pi_w(a) \cap S^\uparrow) \cup (\sigma_w(a) \cap S^\uparrow) \quad (3)$$

We have $\psi(\mathcal{F}_{\bar{a}} \cap S^\uparrow) \subseteq \psi(\mathcal{F}_{\bar{a}}) \cap S^\uparrow$ with equality if $S = \min(\mathcal{F})$, whence $\sum_a |\psi(\mathcal{F}_{\bar{a}}) \cap S^\uparrow| \geq \sum_a |\psi(\mathcal{F}_{\bar{a}} \cap S^\uparrow)| = \sum_{\eta \in \mathcal{F}[S]} (n - |\eta|)$. Thus summing all the equalities (3) on the index $a \in X$, we get

$$2 \sum_{\eta \in \mathcal{F}[S]} |\eta| \geq n |\mathcal{F}[S]| + \sum_{a \in X} |\pi_w(a) \cap S^\uparrow| + |\sigma_w(a) \cap S^\uparrow|. \quad (4)$$

By Theorem 1, $\sigma_w(a) \cap S^\uparrow = \{\varphi_w(f), f \in \mathcal{F}[S], a \in \varphi_w(f) \setminus f\}$, and so:

$$\sum_{a \in X} |\sigma_w(a) \cap S^\uparrow| = \sum_{f \in \mathcal{F}[S]} (|\varphi_w(f)| - |f|)$$

Moreover by Theorem 1 we also get

$$\begin{aligned} \sum_{f \in \mathcal{F}[S]} |f| &= \sum_{\eta \in \mathcal{F}[S]} |\varphi_w^{-1}(\eta)| = \sum_{\eta \in \mathcal{F}[S]} |\eta| - |\eta \setminus \varphi_w^{-1}(\eta)| = \\ &= \sum_{\eta \in \mathcal{F}[S]} |\eta| - \sum_{f \in \mathcal{F}[S]} (|\varphi_w(f)| - |f|) = \sum_{\eta \in \mathcal{F}[S]} |\eta| - \sum_{a \in X} |\sigma_w(a) \cap S^\uparrow| \end{aligned}$$

Therefore using (4) and $\mathcal{F}[S] \simeq \mathcal{F}[S]$ (Corollary 1) we get

$$\sum_{f \in \mathcal{F}[S]} |f| \geq \frac{n}{2} |\mathcal{F}[S]| + \frac{1}{2} \sum_{a \in X} |\pi_w(a) \cap S^\uparrow| - |\sigma_w(a) \cap S^\uparrow|$$

with equality if $S = \min(\mathcal{F})$. \square

We have the following corollary on the local average in the case $\min(\mathcal{F})$ is a maximal antichain and the elements are uniformly bounded by some integer.

Corollary 3. *Let \mathcal{F} be a \cup -closed family of sets such that $\mathcal{G} = \min(\mathcal{F})$ is a maximal antichain of 2^X and there is a positive integer k such that for all $g \in \mathcal{G}$, $|g| \leq k$, then:*

$$\frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} |f| \geq \frac{n-k}{2} + \frac{1}{2|\mathcal{F}[S]|} \sum_{a \in X} |\pi_w(a) \cap S^\uparrow|.$$

Proof. We have already noted in the proof of Proposition 10 that:

$$\sum_{a \in X} |\sigma_w(a) \cap S^\uparrow| = \sum_{f \in \mathcal{F}[S]} (|\varphi_w(f)| - |f|) = \sum_{f \in \mathcal{F}[S]} |\varphi_w(f) \setminus f|$$

by the same proposition it is sufficient to prove that $|\varphi_w(f) \setminus f| \leq k$. Since \mathcal{G} is a maximal antichain, then for any $f \in \mathcal{F}[S]$ there is a $g \in \mathcal{G}$ such that either $g \subseteq \varphi_w(f) \setminus f$ or $\varphi_w(f) \setminus f \subseteq g$. We prove that only $\varphi_w(f) \setminus f \subseteq g$ can occur, and so $|\varphi_w(f) \setminus f| \leq k$. Indeed, if $g \subseteq \varphi_w(f) \setminus f$, then $g \subseteq \varphi_w(f)$ and so, by Theorem 1, $g \subseteq f$, a contradiction. \square

Observe that Corollary 3 also holds if we assume the existence of a maximal antichain $\mathcal{A} \subseteq \mathcal{F}$ such that $|g| \leq k$ for all $g \in \mathcal{A}$.

The following corollary is the analogous of Corollary 3 in the case we drop the maximality condition. Let $S \subseteq \mathcal{F}$ we define $\sigma(S) = \max\{|\sigma_{w\theta}(f)| : f \in S, \theta \in \mathfrak{G}_X\}$.

Corollary 4.

$$\frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} |f| \geq \frac{n - \sigma(S)}{2} + \frac{1}{2|\mathcal{F}[S]|} \sum_{a \in X} |\pi_w(a) \cap S^\uparrow|.$$

Proof. Like in the proof of Corollary 3 and by Proposition 10 it is sufficient to show $|\varphi_w(f) \setminus f| = |\sigma_w(\varphi_w(g))| \leq \sigma(S)$ for all $g \in \mathcal{F}[S]$. Consider any $g \in \mathcal{F}[S]$, and let $f \in S$ such that $f \subseteq g$. By Lemma 9 there is a word w' such that

$$\sigma_w(\varphi_w(g)) = \varphi_w(g) \setminus g = \varphi_{w'}(g) \setminus g \subseteq \varphi_{w'}(f) \setminus f = \sigma_{w'}(\varphi_{w'}(f))$$

and so the claim $|\sigma_w(\varphi_w(g))| \leq \sigma(S)$. \square

The following theorem gives a lower bound of the average localized on S depending on the parameter $|S^\uparrow|$.

Theorem 3.

$$\frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} |f| \geq \frac{n}{2} + \frac{1}{2|\mathcal{F}[S]|} \sum_{a \in X} |\pi_w(a) \cap S^\uparrow| - \frac{1}{2} \log_2 \left\{ \frac{|S^\uparrow|}{|\mathcal{F}[S]|} \right\}$$

and the bound is attained when $S = \min(\mathcal{F})$ and when \mathcal{F} is upward-closed.

Proof. By Proposition 10 it is enough to give an upper bound to the quantity $\sum_{a \in X} |\sigma_w(a) \cap S^\uparrow|$. Following a similar argument in [18], we use Jensen's inequality to upper bound $\sum_{a \in X} |\sigma_w(a) \cap S^\uparrow| = \sum_{f \in \mathcal{F}[S]} (|\varphi_w(f)| - |f|)$. Indeed, we have

$$\exp_2 \left\{ \frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} (|\varphi_w(f)| - |f|) \right\} \leq \frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} 2^{|\varphi_w(f)| - |f|}.$$

By Proposition 2, $f \neq g$ implies $[f, \varphi_w(f)] \cap [g, \varphi_w(g)] = \emptyset$, hence since $||[f, \varphi_w(f)]| = 2^{|\varphi_w(f)| - |f|}$ we get

$$\frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} (|\varphi_w(f)| - |f|) \leq \log_2 \left\{ \frac{|C(\mathcal{F}, S)|}{|\mathcal{F}[S]|} \right\}.$$

where $C(\mathcal{F}, S) = \bigcup_{f \in \mathcal{F}[S]} [f, \varphi_w(f)]$. The statement of the theorem thus follows from $C(\mathcal{F}, S) \subseteq S^\uparrow$.

If $S = \min(\mathcal{F})$ we have the equality in the bound of Proposition 10, moreover if \mathcal{F} is upward-closed, then $\mathcal{F} = \mathcal{F}$. Thus $\sum_{a \in X} |\sigma_w(a) \cap S^\uparrow| = 0$ which is equal to $\frac{1}{2} \log_2 \{|S^\uparrow|/|\mathcal{F}[S]|\}$ since \mathcal{F} is upward-closed and so $S^\uparrow = \mathcal{F}[S]$. Therefore the lower bound in the statement is reached for $S = \min(\mathcal{F})$ and the class of upward-closed families. \square

Remark 1. In Theorem 3, Corollaries 4,3, we can give a lower bound to the quantity $\frac{1}{2^{|\mathcal{F}[S]|}} \sum_{a \in X} |\pi_w(a) \cap S^\uparrow|$. Indeed, by Proposition 8 and Theorem 1 it is not difficult to see that

$$\sum_{a \in X} |\pi_w(a) \cap S^\uparrow| \geq \sum_{g \in \mathcal{F}[S]} |\{a \in g : (g \setminus \{a\})^* = \emptyset\}|$$

and the equality is attained if $S = \min(\mathcal{F})$ and when \mathcal{F} is upward-closed.

5.3 The invariant case

In this section we obtain some lower bounds on the average of \mathcal{F} localized on S using the invariant upward-closed set associated to \mathcal{F} . The following definition can be considered as the analogous of the spurious and pure elements of Definition 2 in the invariant case.

Definition 3. Let $\mathbf{U}(\mathcal{F})$ be the invariant upward-closed set associated to \mathcal{F} and let $a \in X$ the set

$$\Sigma(\mathcal{F}, a) = \{g \in \mathcal{F}_a : \forall \eta \in \max(\text{Fib}(g)), a \in \eta\}$$

is called the set of hyper-spurious elements. The local version of this set is $\Sigma(\mathcal{F}, g) = \{a \in X \setminus g : \forall \eta \in \max(\text{Fib}(g)), a \in \eta\}$. The elements of the set

$$\Pi(\mathcal{F}, a) = \{g \in \mathcal{F}_a : \forall \eta \in \max(\text{Fib}(g)), \eta \setminus \{a\} \notin \mathbf{U}(\mathcal{F})\}$$

are called hyper-pure. The local version of this set is $\Pi(\mathcal{F}, g) = \{a \in g : \forall \eta \in \max(\text{Fib}(g)), \eta \setminus \{a\} \notin \mathbf{U}(\mathcal{F})\}$.

The connection between these sets and the spurious, pure sets introduced in Definition 2 is given by the following proposition.

Proposition 11. The following equalities hold:

$$\Sigma(\mathcal{F}, g) = \bigcap_{\eta \in \max(\text{Fib}(g))} \eta \setminus g = \bigcap_{\theta \in \mathfrak{S}_X} \sigma_{w\theta}(\mathcal{F}, \varphi_{w\theta}(g)) \quad (5)$$

$$\Pi(\mathcal{F}, g) = \bigcap_{\theta \in \mathfrak{S}_X} \pi_{w\theta}(\mathcal{F}, \varphi_{w\theta}(g)) \quad (6)$$

Proof. The first equality in (5) is a consequence of the definition, the second one of Theorem 2. Let us prove (6). Let $b \in \Pi(\mathcal{F}, g)$, then for any $\eta \in \max(\text{Fib}(g)) = \mathfrak{S}_X \cdot \varphi_w(g)$ (by Theorem 2) we have $\eta \setminus \{b\} \notin \mathbf{U}(\mathcal{F})$, hence for any $\theta \in \mathfrak{S}_X$, $\varphi_{w\theta}(g) \setminus \{b\} \notin \varphi_{w\theta}(\mathcal{F})$, i.e. $b \in \bigcap_{\theta \in \mathfrak{S}_X} \pi_{w\theta}(\mathcal{F}, \varphi_{w\theta}(g))$. On the other side, let $b \in \bigcap_{\theta \in \mathfrak{S}_X} \pi_{w\theta}(\mathcal{F}, \varphi_{w\theta}(g))$. To obtain a contradiction suppose that there is $\eta \in \max(\text{Fib}(g))$, for some $g \in \mathcal{F}$, such that $\eta \setminus \{b\} \in \mathbf{U}(\mathcal{F})$, say $\eta \setminus \{b\} \in \max(\text{Fib}(h))$ for some $h \in \mathcal{F}$. Since by Theorem 2 $\max(\text{Fib}(h)) = \mathfrak{S}_X \cdot \varphi_w(h)$, there is a $\vartheta \in \mathfrak{S}_X$ such that $\eta \setminus \{b\} = \varphi_{w\vartheta}(h)$. Since $\eta \setminus \{b\} \subseteq \eta$ we have $\eta \in \varphi_{w\vartheta}(\mathcal{F})$, in particular, since $\varphi_{w\vartheta}^{-1}(\eta) = \eta^* = g$, we have $\eta = \varphi_{w\vartheta}(g)$. However $b \in \bigcap_{\theta \in \mathfrak{S}_X} \pi_{w\theta}(\mathcal{F}, \varphi_{w\theta}(g))$ implies $b \in \pi_{w\vartheta}(\mathcal{F}, \eta)$ which contradicts $\eta \setminus \{b\} \in \varphi_{w\vartheta}(\mathcal{F})$. \square

We recall that at the end of Section 4 we have introduced the set

$$\text{Cov}_a(g) = \{h \in \mathcal{F}_a : (h \setminus \{a\})^* = g\}$$

we have the following proposition:

Proposition 12.

$$\Sigma(\mathcal{F}, g) = \{b \in X \setminus g : \text{Cov}_b(g) = \emptyset\} = X \setminus \bigcup_{\{h: g < h\}} h$$

$$|\mathcal{F}_{\bar{a}}| - |\Sigma(\mathcal{F}, a)| \leq \sum_{g \in \mathcal{F}_{\bar{a}} \setminus \Sigma(\mathcal{F}, a)} |\max(\text{Cov}_a(g))| \leq |\mathcal{F}_a| - |\Pi(\mathcal{F}, a)|, \forall a \in X$$

Proof. By Proposition 5 and Lemma 7 we have that $\text{Cov}_a(g) = \emptyset$ if and only if $a \in \Sigma(\mathcal{F}, g)$. The second equality is also a consequence of Proposition 5 and the definitions. Let us prove the inequalities. We first claim that for any $h \in \max(\text{Cov}_a(g))$ with $g \in \mathcal{F}_{\bar{a}} \setminus \Sigma(\mathcal{F}, a)$ and for any $\eta \in \max(\text{Fib}(h))$, $\eta \setminus \{a\} \in \mathbf{U}(\mathcal{F})$. Since $(h \setminus \{a\})^* = g$ and $(h \setminus \{a\}) \subseteq \eta \setminus \{a\}$ we have $g \subseteq (\eta \setminus \{a\})^*$. On the other hand, let $(\eta \setminus \{a\})^* = g'$. Since $g' \subseteq \eta \setminus \{a\}$ and $g' \subseteq \eta^* = h$, then $g' \subseteq (h \setminus \{a\})^* = g$ from which we have the equality $(\eta \setminus \{a\})^* = g$. Therefore $(\eta \setminus \{a\}) \in \text{Fib}(g)$, and so there is a $\nu \in \max(\text{Fib}(g))$ with $(\eta \setminus \{a\}) \subseteq \nu$. Consider $\nu \cup \{a\}$ and let us prove that $\nu \cup \{a\} \in \max(\text{Fib}(h))$. Clearly $\nu \cup \{a\} \in \max(\text{Fib}(h'))$ for some h' , we observe that since $h \subseteq \eta \subseteq \nu \cup \{a\}$ we have $h \subseteq (\nu \cup \{a\})^* = h'$. If we prove that $h' \in \text{Cov}_a(g)$, then by the maximality of h , we get $h = h'$. Suppose, contrary to our claim, that $h' \notin \text{Cov}_a(g)$. Thus, if we put $g' = (h' \setminus \{a\})^*$,

we have $g = (h \setminus \{a\})^* \subsetneq (h' \setminus \{a\})^* = g'$. However, we also have $g' = (h' \setminus \{a\})^* \subseteq \nu^* = g$, a contradiction. Therefore, the claim is true and so we can deduce the following inclusion:

$$\bigcup_{g \in \mathcal{F}_{\bar{a}} \setminus \Sigma(\mathcal{F}, a)} \max(\text{Cov}_a(g)) \subseteq \mathcal{F}_a \setminus \Pi(\mathcal{F}, a)$$

Thus, the inequality easily follows from this inclusion and the following facts: $\text{Cov}_a(g) \neq \emptyset$ iff $g \in \mathcal{F}_{\bar{a}} \setminus \Sigma(\mathcal{F}, a)$ and $\text{Cov}_a(g) \cap \text{Cov}_a(g') = \emptyset$ for $g \neq g'$. \square

The following theorem is the analogous of Theorem 3 in the invariant case.

Proposition 13.

$$\frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} |f| \geq \frac{n}{2} - \frac{1}{2|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} |\Sigma(\mathcal{F}, g)| + \frac{1}{2|\mathcal{F}[S]|} \sum_{a \in X} |\Pi(\mathcal{F}, a) \cap S^\uparrow|$$

this bound is attained for $S = \min(\mathcal{F})$ and when \mathcal{F} is upward-closed.

Proof. Proposition 12 can be easily adapt to prove that for all $a \in X$:

$$|\mathcal{F}_{\bar{a}}[S]| - |\Sigma(\mathcal{F}, a) \cap S^\uparrow| \leq |\mathcal{F}_a[S]| - |\Pi(\mathcal{F}, a) \cap S^\uparrow|$$

The statement can be thus proved summing all these inequalities with a running on X and using the equalities $\sum_{a \in X} |\mathcal{F}_a[S]| = \sum_{f \in \mathcal{F}_a[S]} |f|$, $\sum_{a \in X} |\mathcal{F}_{\bar{a}}[S]| = \sum_{f \in \mathcal{F}_{\bar{a}}[S]} (n - |f|)$, $\sum_{a \in X} |\Sigma(\mathcal{F}, a) \cap S^\uparrow| = \sum_{f \in \mathcal{F}_a[S]} |\Sigma(\mathcal{F}, f)|$. By Theorem 3 and Remark 1 in the case of an upward-closed family \mathcal{F} and $S = \min(\mathcal{F})$, we have

$$\frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} |f| = \frac{n}{2} + \frac{1}{2|\mathcal{F}[S]|} \sum_{g \in \mathcal{F}[S]} |\{a \in g : (g \setminus \{a\})^* = \emptyset\}|$$

On the other hand by Proposition 12 it is not difficult to check that in the case \mathcal{F} is an upward-closed family $\Sigma(\mathcal{F}, g) = \emptyset$ and $\Pi(\mathcal{F}, g) = \{a \in g : (g \setminus \{a\})^* = \emptyset\}$ and so the bound is attained in this case. \square

Using the first equality of Proposition 12 and Proposition 13 we can rewrite the bound of Proposition 13 as

$$\frac{1}{|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} |f| \geq \frac{1}{2|\mathcal{F}[S]|} \sum_{f \in \mathcal{F}[S]} \left| \bigcup_{\{h: f \triangleleft h\}} h \right| + \frac{1}{2|\mathcal{F}[S]|} \sum_{a \in X} |\Pi(\mathcal{F}, a) \cap S^\uparrow|$$

We observe that by Propositions 11, 8, similarly to Remark 1, we also have the following lower bound

$$\sum_{a \in X} |\Pi(\mathcal{F}, a) \cap S^\uparrow| \geq \sum_{g \in \mathcal{F}[S]} |\{a \in g : (g \setminus \{a\})^* = \emptyset\}|$$

6 Upper bounds for the join-irreducible elements of a union-closed family

Let \mathcal{F} be a \cup -closed family of sets of 2^X with $X = \{a_1, a_2, \dots, a_n\}$, in this section we use the techniques obtained in Section 3 to give an upper bound to the number of join-irreducible elements of \mathcal{F} . We remark that if $m \in J(\mathcal{F})$ then $\mathcal{F} \setminus \{m\}$ is again a \cup -closed family of 2^X . Therefore it is interesting and quite natural studying the effect of erasing an irreducible elements from \mathcal{F} in the rising process. For this reasons we will denote by φ_w, φ'_w the rising function with respect to the word $w = a_1 a_2 \dots a_n$ respectively of $\mathcal{F}, \mathcal{F}' = \mathcal{F} \setminus \{m\}$. We recall that the rising function at the i -th step is defined by

$$\varphi_{\mathcal{F}_i, a_{i+1}}(z) = \begin{cases} z \cup \{a_{i+1}\} & \text{if } z \cup \{a_{i+1}\} \notin \mathcal{F}_i, \\ z & \text{otherwise;} \end{cases}$$

Here we simplify the cumbersome notation and we write $\varphi_{a_{i+1}}, \varphi'_{a_{i+1}}$ for $\varphi_{\mathcal{F}_i, a_{i+1}}, \varphi'_{\mathcal{F}'_i, a_{i+1}}$, respectively. With this notation, the rising function with respect to $w = a_1 \dots a_n$ is the last function φ_n of the sequence of functions defined inductively by $\varphi_i = \varphi_{a_i} \circ \varphi_{i-1}$ for $i = 1, \dots, n$ where φ_0 is the identity function on 2^X . We have the following lemma.

Lemma 10. *With the above notation, for each $i \in \{0, 1, \dots, n\}$ there is a element $\mu_i \in \mathcal{F}_i$ such that $\mathcal{F}'_i = \mathcal{F}_i \setminus \{\mu_i\}$. Moreover we have two possibilities*

1. *if there is no $z \in \mathcal{F}'_i$ such that $z \cup a_{i+1} = \mu_i$, then $\mathcal{F}'_{i+1} = \mathcal{F}_{i+1} \setminus \{\mu_{i+1}\}$ with $\mu_{i+1} = \varphi_{a_{i+1}}(\mu_i)$ and $\varphi_{a_{i+1}}(z) = \varphi'_{a_{i+1}}(z)$ for all $z \in \mathcal{F}'_i$.*
2. *if there is $z \in \mathcal{F}'_i$ such that $z \cup a_{i+1} = \mu_i$, then $\mathcal{F}'_{i+1} = \mathcal{F}_{i+1} \setminus \{\mu_{i+1}\}$ with $\mu_{i+1} = z = \varphi_{a_{i+1}}(z)$, $\varphi'_{a_{i+1}}(z) = \mu_i$ and $\varphi'_{a_{i+1}}(y) = \varphi_{a_{i+1}}(y)$ for all $y \in \mathcal{F}'_i \setminus \{z\}$.*

Proof. We prove the statement by induction on the index i . The statement is true for $i = 0$, since $\mathcal{F}'_0 = \mathcal{F}' = \mathcal{F} \setminus \{m\} = \mathcal{F}_0 \setminus \{m\}$. So, putting $\mu_0 = m$, we can suppose that the statement is true for $i > 0$ and let us prove it for $i + 1$. By induction, there is an element $\mu_i \in \mathcal{F}_i$ such that $\mathcal{F}'_i = \mathcal{F}_i \setminus \{\mu_i\}$. Let $z \in \mathcal{F}'_i$, we have the following cases:

- i) $z \cup \{a_{i+1}\} \notin \mathcal{F}_i$ and so also $z \cup \{a_{i+1}\} \notin \mathcal{F}'_i$ which implies $\varphi_{a_{i+1}}(z) = \varphi'_{a_{i+1}}(z) = z \cup \{a_{i+1}\}$.
- ii) $z \cup \{a_{i+1}\} \in \mathcal{F}'_i \subseteq \mathcal{F}_i$ and so $\varphi_{a_{i+1}}(z) = \varphi'_{a_{i+1}}(z) = z$.

iii) $z \cup \{a_{i+1}\} \in \mathcal{F}_i \setminus \mathcal{F}'_i = \{\mu_i\}$, and so $z \cup \{a_{i+1}\} = \mu_i$. Hence $\varphi'_{a_{i+1}}(z) = z \cup \{a_{i+1}\} = \mu_i = \varphi_{a_{i+1}}(\mu_i)$ and $\varphi_{a_{i+1}}(z) = z$.

Thus, if condition $z \cup \{a_{i+1}\} = \mu_i$ do not hold for any $z \in \mathcal{F}_i$, then i), ii) hold and so condition 1. is true. Otherwise if there is $z \in \mathcal{F}'_i$ such that $z \cup a_{i+1} = \mu_i$, then iii) holds and so $z = \varphi_{a_{i+1}}(z)$ is missing in \mathcal{F}'_{i+1} , whence $\mathcal{F}'_{i+1} = \mathcal{F}_{i+1} \setminus \{\mu_{i+1}\}$ with $\mu_{i+1} = z$ and $\varphi'_{a_{i+1}}(z) = \mu_i$. For any $y \in \mathcal{F}'_i \setminus \{z\}$ either condition i) or ii) holds and so $\varphi'_{a_{i+1}}(y) = \varphi_{a_{i+1}}(y)$, and this concludes the proof of statement 2. \square

The previous Lemma shows that in each i -section \mathcal{F}'_i there is exactly one missing element belonging to $\mathcal{F}_i \setminus \mathcal{F}'_i$, this element plays an important role in the way the raising function changes. For this reason we call μ_i of Lemma 10, the *missing element* at the i -th section. The next lemma gives a more precise description of the way the rising function changes.

Lemma 11 (swapping lemma). *With the notation of Lemma 10, there are $k + 1$ different elements $m_i \in \mathcal{F}$, for $i = 0, \dots, k$ such that $m_0 = m$ and an increasing sequence of k integers $1 \leq i_1 < \dots < i_k < n$ such that for all $0 < j \leq k$*

$$\varphi'_t(m_j) = \begin{cases} \varphi_t(m_j) & \text{if } t < i_j \\ \varphi_t(m_{j-1}) & \text{otherwise;} \end{cases}$$

while $\varphi'_t(z) = \varphi_t(z)$ for all $1 \leq t \leq n$ and $z \in \mathcal{F} \setminus \{m_0, \dots, m_k\}$. For any $0 \leq i \leq n$ the missing element is $\mu_i = \varphi_i(m_s)$ where $0 < s \leq k$ satisfies $i_s \leq i < i_{s+1}$ if $s < k$ or $i_k \leq i < n$ if $s = k$. Moreover for all $1 \leq j \leq k$, $\varphi'_{i_j-1}(m_j) \cup \{a_{i_j}\} = \varphi_{i_j-1}(m_{j-1})$.

Proof. By Lemma 10 we have $\varphi'_{a_t}(z) = \varphi_{a_t}(z)$ for all $z \in \mathcal{F}$ and the missing element is $\mu_t = \varphi_t(m_0)$ for all $1 \leq t < i_1 \leq n$ where i_1 is the first integer such that there is an element $\varphi'_{i_1-1}(m_1) \in \mathcal{F}'_{i_1-1}$, for some $m_1 \in \mathcal{F}$ with $m_1 \neq m_0$, satisfying $\varphi'_{i_1-1}(m_1) \cup \{a_{i_1}\} = \mu_{i_1-1} = \varphi_{i_1-1}(m_0)$. Therefore by Lemma 10 we have that $\varphi'_{i_1}(m_1) = \varphi_{i_1-1}(m_0) = \varphi_{i_1}(m_0)$ and the missing element becomes $\mu_{i_1} = \varphi_{i_1}(m_1)$. Moreover, since $\varphi'_{i_1-1}(m_1) = \varphi_{i_1-1}(m_1)$ and $\varphi'_{i_1-1}(m_1) \cup \{a_{i_1}\} = \mu_{i_1-1} = \varphi_{i_1-1}(m_0) \in \mathcal{F}_{i_1-1}$, by Lemma 1 we get $a_{i_1} \notin \varphi_{i_1}(m_1) = \mu_{i_1}$. In this way we have proved the base case of the following property

P_h : There is a sequence of integers $1 \leq i_1 < \dots < i_j \leq h < n$ and $j + 1$ different elements $m_0, \dots, m_j \in \mathcal{F}$ such that for all $0 < l \leq j$ and for all $t \leq h$

$$\varphi'_t(m_l) = \begin{cases} \varphi_t(m_l) & \text{if } t < i_l \\ \varphi_t(m_{l-1}) & \text{otherwise;} \end{cases}$$

$\varphi'_t(z) = \varphi_t(z)$ for all $1 \leq t \leq h$ and $z \in \mathcal{F} \setminus \{m_0, \dots, m_j\}$. For all $0 \leq i \leq n$, $\mu_i = \varphi_i(m_s)$ where $0 < s \leq j$ satisfies $i_s \leq i < i_{s+1}$, $\mu_i = \varphi_i(m_j)$ for $i_j \leq i \leq h$. Moreover for all $0 < s \leq j$, $\varphi'_{i_s-1}(m_s) \cup \{a_{i_s}\} = \varphi_{i_s-1}(m_{s-1})$ and $a_{i_1}, \dots, a_{i_s} \notin \mu_{i_s}$.

Let us prove this property by induction. By Lemma 10 it is clear that if for any $z \in \mathcal{F}_h$ the condition $z \cup \{a_{h+1}\} = \mu_h$ does not occur, then P_{h+1} is true. Suppose that $z \cup \{a_{h+1}\} = \mu_h$. Let us prove that there is an element $m_{j+1} \in \mathcal{F}$ different from m_l for all $l \leq j$ such that $z = \varphi'_h(m_{j+1})$. Suppose, contrary to our claim, that $m_{j+1} = m_s$ for some $s \leq j$. We first claim that $a_{i_s} \in \varphi'_h(m_s) \setminus \mu_h$. Indeed conditions $a_{i_1}, \dots, a_{i_j} \notin \mu_{i_j}$ and $s \leq j$ yields to $a_{i_s} \notin \mu_{i_j} = \varphi_{i_j}(m_j)$ and so, since $\mu_h = \varphi_h(m_j)$ and $i_s \leq i_j \leq h$, we get $a_{i_s} \notin \mu_h$. Since $\varphi'_{i_s-1}(m_s) \cup \{a_{i_s}\} = \varphi_{i_s-1}(m_{s-1})$, by Lemma 1 $a_{i_s} \in \varphi_{i_s-1}(m_{s-1}) = \varphi_{i_s}(m_{s-1})$, hence by property P_h , we get $a_{i_s} \in \varphi_{i_s-1}(m_{s-1}) = \varphi_{i_s}(m_{s-1}) = \varphi'_{i_s}(m_s)$. Thus $a_{i_s} \in \varphi'_h(m_s)$ and so, with $a_{i_s} \notin \mu_h$, we get the claim $a_{i_s} \in \varphi'_h(m_s) \setminus \mu_h$. However this contradicts $\varphi'_h(m_s) \cup \{a_{h+1}\} = z \cup \{a_{h+1}\} = \mu_h$.

Therefore we can suppose that there is a $m_{j+1} \in \mathcal{F}$ different from m_0, \dots, m_j such that $\varphi'_h(m_{j+1}) \cup \{a_{h+1}\} = \mu_h$. Therefore by induction we get $\varphi'_t(m_{j+1}) = \varphi_t(m_{j+1})$ for all $t \leq h = i_{j+1} - 1$. Putting $i_{j+1} = h + 1$ we get, by Lemma 10 and P_h

$$\varphi'_{i_{j+1}}(m_{j+1}) = \mu_{i_{j+1}-1} = \mu_h = \varphi_h(m_j) = \varphi_{h+1}(m_j) = \varphi_{i_{j+1}}(m_j)$$

since $a_{h+1} \in \mu_h$ and so $\mu_h = \varphi_h(m_j) = \varphi_{h+1}(m_j)$. Moreover we also have $\varphi'_{i_{j+1}-1}(m_{j+1}) \cup \{a_{i_{j+1}}\} = \varphi_{i_{j+1}-1}(m_j)$.

Since $\varphi'_h(m_{j+1}) \cup \{a_{h+1}\} = \mu_h \in \mathcal{F}_h$ and $\varphi'_h(m_{j+1}) = \varphi_h(m_{j+1})$ we have that $\varphi_h(m_{j+1}) = \varphi_{h+1}(m_{j+1})$ and so by Lemma 10 the missing element becomes

$$\mu_{i_{j+1}} = \varphi'_h(m_{j+1}) = \varphi_h(m_{j+1}) = \varphi_{h+1}(m_{j+1}) = \varphi_{i_{j+1}}(m_{j+1})$$

Hence $\mu_{i_{j+1}} \cup \{a_{h+1}\} = \mu_h$ and so, by Lemma 1, $a_{h+1} \notin \mu_{i_{j+1}}$. To conclude the proof we have to show $a_{i_1}, \dots, a_{i_{j+1}} \notin \mu_{i_{j+1}}$. By induction $a_{i_1}, \dots, a_{i_j} \notin \mu_{i_j}$, hence $a_{i_1}, \dots, a_{i_j} \notin \mu_h$. Therefore, from $\mu_{i_{j+1}} \cup \{a_{h+1}\} = \mu_h$ and $a_{h+1} \notin \mu_{i_{j+1}}$ we get $a_{i_1}, \dots, a_{i_j}, a_{i_{j+1}} \notin \mu_{i_{j+1}}$. \square

We remark that the above swapping Lemma holds for a general family of subsets \mathcal{F} since the hypothesis of \cup -closure is never used in the proof. As a consequence of the previous swapping Lemma we have the following proposition.

Proposition 14. Let \mathcal{F} be a family of subsets of $X = \{a_1, \dots, a_n\}$ and let $m \in \mathcal{F}$. Consider the rising functions φ_w, φ'_w with respect to the word $w = a_1 a_2 \dots a_n$ respectively of $\mathcal{F}, \mathcal{F}' = \mathcal{F} \setminus \{m\}$. There are $k+1$ different elements $m_i \in \mathcal{F}$, for $i = 0, \dots, k$ such that $m_0 = m$ and for all $z \in \mathcal{F} \setminus \{m_0, \dots, m_k\}$ we have $\varphi'_w(z) = \varphi_w(z)$ while for all $0 < j \leq k$

$$\varphi'_w(m_j) = \varphi_w(m_{j-1})$$

in particular $\varphi'_w(\mathcal{F}') = \varphi_w(\mathcal{F}) \setminus \{\varphi_w(m_k)\}$ and $\varphi_w(m_k) \in \min(\varphi_w(\mathcal{F}))$. Moreover $a_{i_j} \in m_{j-1} \setminus \varphi_w(m_j)$ for all $1 \leq j \leq k$.

Proof. The first claim is an immediate consequence of Lemma 11 when $t = n$. In particular the missing element $\mu_n = \varphi_w(m_k)$ and so $\varphi'_w(\mathcal{F}') = \varphi_w(\mathcal{F}) \setminus \{\varphi_w(m_k)\}$. Moreover since both $\varphi'_w(\mathcal{F}'), \varphi_w(\mathcal{F})$ are upward-closed sets, then it is straightforward to prove that necessarily the missing elements must be minimal, otherwise $\varphi_w(\mathcal{F}) \setminus \{\varphi_w(m_k)\}$ would not be upward-closed, whence $\varphi_w(m_k) \in \min(\varphi_w(\mathcal{F}))$.

From Lemma 11 we have that for all $1 \leq j \leq k$, $\varphi'_{i_j-1}(m_j) \cup \{a_{i_j}\} = \varphi_{i_j-1}(m_{j-1})$ and $\varphi'_{i_j-1}(m_j) = \varphi_{i_j-1}(m_j)$, whence

$$\varphi_{i_j-1}(m_j) \cup \{a_{i_j}\} = \varphi_{i_j-1}(m_{j-1})$$

and so, by Lemma 1, we get for all $1 \leq j \leq k$, $a_{i_j} \in m_{j-1} \setminus \varphi_w(m_j)$. \square

We now assume \mathcal{F} \cup -closed and we consider the situation when we take away an irreducible element $m \in J(\mathcal{F})$. In this case we have a limitation on the number of possible swappings, indeed the following proposition holds.

Proposition 15. Let \mathcal{F} be a \cup -closed family of subsets of a set $X = \{a_1, \dots, a_n\}$ and let $m \in J(\mathcal{F})$. Consider the rising functions φ_w, φ'_w with respect to the word $w = a_1 a_2 \dots a_n$ respectively of $\mathcal{F}, \mathcal{F}' = \mathcal{F} \setminus \{m\}$ and denote by $\mathcal{F} = \varphi_w(\mathcal{F}), \mathcal{F}' = \varphi'_w(\mathcal{F}')$. There are two possibilities:

1. $\varphi_w(m) \in \min(\mathcal{F})$, $\mathcal{F}' = \mathcal{F} \setminus \{\varphi_w(m)\}$ and for all $z \in \mathcal{F}'$ $\varphi'_w(z) = \varphi_w(z)$.
2. The set $\overline{m} = \cup_{\{f \in \mathcal{F} : f \subsetneq m\}} f$ is non-empty. For all $z \in \mathcal{F} \setminus \{m, \overline{m}\}$ we have $\varphi'_w(z) = \varphi_w(z)$ and $\varphi'_w(\overline{m}) = \varphi_w(m)$. Moreover $\varphi_w(\overline{m}) \in \min(\mathcal{F})$ and $\mathcal{F}' = \mathcal{F} \setminus \{\varphi_w(\overline{m})\}$.

Proof. Using the notation of Proposition 14, suppose $k \geq 2$. Therefore, there are two distinct elements m_1, m_2 different from m such that $\varphi'_w(m_2) = \varphi_w(m_1)$ and $\varphi'_w(m_1) = \varphi_w(m)$. We claim

$$\{f \in \mathcal{F} : f \subseteq \varphi_w(m_1)\} = \{f \in \mathcal{F}' : f \subseteq \varphi_w(m_1)\} \quad (7)$$

Clearly $\{f \in \mathcal{F}' : f \subseteq \varphi_w(m_1)\} \subseteq \{f \in \mathcal{F} : f \subseteq \varphi_w(m_1)\}$ and to prove the other inclusion it is sufficient to prove that $m \not\subseteq \varphi_w(m_1)$. Suppose on the contrary that actually $m \subseteq \varphi_w(m_1)$, however by Proposition 14, $a_{i_1} \in m \setminus \varphi_w(m_1)$, a contradiction. Thus (7) holds. Since $m \in J(\mathcal{F})$, then $\mathcal{F}' = \mathcal{F} \setminus \{m\}$ is a \cup -closed family and so by Corollary 1, equality (7) and $\varphi'_w(m_2) = \varphi_w(m_1)$ we get

$$\begin{aligned} m_1 &= \varphi_w^{-1}(\varphi_w(m_1)) = \bigcup_{\{f \in \mathcal{F} : f \subseteq \varphi_w(m_1)\}} f = \bigcup_{\{f \in \mathcal{F}' : f \subseteq \varphi_w(m_1)\}} f \\ &= \bigcup_{\{f \in \mathcal{F}' : f \subseteq \varphi'_w(m_2)\}} f = \varphi'^{-1}_w(\varphi'_w(m_2)) = m_2 \end{aligned}$$

a contradiction. Therefore we have two possibilities either $k = 0$ or $k = 1$. Applying Proposition 14 to the case $k = 0$ we get for all $z \in \mathcal{F} \setminus \{m\}$ $\varphi'_w(z) = \varphi_w(z)$ and $\mathcal{F}' = \mathcal{F} \setminus \{\varphi_w(m)\}$ and $\varphi_w(m) \in \min(\mathcal{F})$. Consider the case $k = 1$. We prove that in this case $m_1 = \overline{m}$ where $\overline{m} = \cup_{\{f \in \mathcal{F} : f \subsetneq m\}} f$. Since $\varphi'_w(m_1) = \varphi_w(m)$ then $m_1 \subsetneq \varphi_w(m)$ and so by Theorem 1 $m_1 \subsetneq m$, hence $\overline{m} \neq \emptyset$. Since $\varphi'_w(m_1) = \varphi_w(m)$ then

$$\{f \in \mathcal{F}' : f \subseteq \varphi'_w(m_1)\} = \{f \in \mathcal{F}' : f \subseteq \varphi_w(m)\} \quad (8)$$

moreover $\{f \in \mathcal{F} : f \subsetneq m\} \subseteq \{f \in \mathcal{F}' : f \subseteq \varphi_w(m)\}$ and by Theorem 1 it is not difficult to check that $\{f \in \mathcal{F}' : f \subseteq \varphi_w(m)\} \subseteq \{f \in \mathcal{F} : f \subsetneq m\}$ also holds. Hence by equality (8) we have $\{f \in \mathcal{F}' : f \subseteq \varphi'_w(m_1)\} = \{f \in \mathcal{F} : f \subsetneq m\}$ and so by Corollary 1

$$m_1 = \varphi'^{-1}_w(\varphi'_w(m_1)) = \bigcup_{\{f \in \mathcal{F}' : f \subseteq \varphi'_w(m_1)\}} f = \bigcup_{\{f \in \mathcal{F} : f \subsetneq m\}} f = \overline{m}$$

The other properties are consequences of Proposition 14. \square

We have the following theorem.

Theorem 4. *Let \mathcal{F} be a \cup -closed family of sets of 2^X with $X = \{a_1, \dots, a_n\}$. Consider the rising function φ_w with respect to the word $w = a_1 a_2 \dots a_n$ and let $\mathcal{F} = \varphi_w(\mathcal{F})$, then*

$$|J(\mathcal{F})| \leq 2|\min(\mathcal{F})| + |\min(\mathcal{F} \setminus \min(\mathcal{F}))|$$

Proof. $J(\mathcal{F})$ can be partitioned into two subsets J_1, J_2 respectively of the elements $m \in J(\mathcal{F})$ such that $\varphi_w(m) \in \min(\mathcal{F})$ and the elements m for which condition 2 of Proposition 15 holds but $\varphi_w(m) \notin \min(\mathcal{F})$ (conditions

1 and 2 of Proposition 15 are not mutually exclusive). Since φ_w is an injection we immediately have

$$|J_1| \leq |\min(\mathcal{F})| \quad (9)$$

We define the partial function $\iota_{\mathcal{F}} : J(\mathcal{F}) \rightarrow \mathcal{F}$ taking an element m into

$$\iota_{\mathcal{F}}(m) = \bigcup_{f \in \mathcal{F} : f \subsetneq m} f$$

It is straightforward to check that whenever it is defined: $\iota_{\mathcal{F}}(m) \subsetneq m$ (it can not be equal since m is irreducible) and if $m' \subsetneq m$, then $m' \subseteq \iota_{\mathcal{F}}(m)$. In view of Proposition 15, we consider the restriction $\iota_{\mathcal{F}} : J_2 \rightarrow \mathcal{F}$ which is a function. Thus for a $\overline{m} \in \iota_{\mathcal{F}}(J_2)$, the set $\iota_{\mathcal{F}}^{-1}(\overline{m})$ is clearly non-empty and let $\iota_{\mathcal{F}}^{-1}(\overline{m}) = \{m_1, \dots, m_k\}$ for some $k \geq 1$. We observe that for all $i \neq j$, $m_i \not\subseteq m_j$ since, otherwise $m_i \subsetneq m_j$ would imply the contradiction $\overline{m} \subsetneq m_i \subseteq \iota_{\mathcal{F}}(m_j) = \overline{m}$. Therefore for all $i \neq j$

$$\iota_{\mathcal{F}}(m_j) = \iota_{\mathcal{F} \setminus \{m_i\}}(m_j) \quad (10)$$

We claim that for all $i \neq j$ we have that at least one between $\varphi_w(m_i), \varphi_w(m_j)$ is minimal in $\mathcal{F}' = \mathcal{F} \setminus \{\varphi_w(\overline{m})\}$. This is a consequence of the application of Proposition 15 twice. Indeed, consider $\mathcal{F} \setminus \{m_i\}$ and let φ'_w be the rising function of this set with respect to w . By Proposition 15 we have $\varphi_w(\overline{m}) \in \min(\mathcal{F})$, $\varphi'_w(\overline{m}) = \varphi_w(m_i)$ and $\varphi'_w(m_j) = \varphi_w(m_j)$. It is evident that $m_j \in J(\mathcal{F} \setminus \{m_i\})$ and so consider the \cup -closed set $(\mathcal{F} \setminus \{m_i\}) \setminus \{m_j\}$. Let φ''_w be the rising function of this set with respect to w . By Proposition 15 we have two possibilities: either $\varphi'_w(m_j) = \varphi_w(m_j)$ is minimal in $\mathcal{F} \setminus \{\varphi_w(\overline{m})\}$, or by (10), we have that

$$\varphi'_w(\iota_{\mathcal{F} \setminus \{m_i\}}(m_j)) = \varphi'_w(\iota_{\mathcal{F}}(m_j)) = \varphi'_w(\overline{m}) = \varphi_w(m_i)$$

is minimal in $\mathcal{F} \setminus \{\varphi_w(\overline{m})\}$. Therefore, it is straightforward to prove that all the m_i except at most one, say m_k , are minimal in $\mathcal{F} \setminus \{\varphi_w(\overline{m})\}$. Hence, denoting by J'_2 the set of elements $m \in J_2$ such that $\varphi_w(m)$ is minimal in $\mathcal{F} \setminus \{\varphi_w(\iota_{\mathcal{F}}(m))\}$, we get that there is an injection of $J_2 \setminus J'_2$ into $\iota_{\mathcal{F}}(J_2 \setminus J'_2)$ which is in one to one correspondence with the elements of $\varphi_w(\iota_{\mathcal{F}}(J_2 \setminus J'_2))$ (being φ_w injective) which is in turn a subset of $\min(\mathcal{F})$ (by definition of the set J_2 and Proposition 15), whence:

$$|J_2 \setminus J'_2| \leq |\min(\mathcal{F})| \quad (11)$$

We now prove that $\varphi_w(J'_2) \subseteq \min(\mathcal{F} \setminus \min(\mathcal{F}))$. Since $J'_2 \subseteq J_2$, then, by definition of J_2 , we have that $\varphi_w(m) \notin \min(\mathcal{F})$ for all $m \in J'_2$. Thus $\varphi_w(J'_2) \subseteq \mathcal{F} \setminus \min(\mathcal{F})$. Moreover, if $m \in J'_2$, then $\varphi_w(m)$ is minimal in $\mathcal{F} \setminus \{\varphi_w(\iota_{\mathcal{F}}(m))\}$ and since $\varphi_w(\iota_{\mathcal{F}}(m)) \in \min(\mathcal{F})$ we have

$$\mathcal{F} \setminus \min(\mathcal{F}) \subseteq \mathcal{F} \setminus \{\varphi_w(\iota_{\mathcal{F}}(m))\}$$

hence $\varphi_w(m)$ is also minimal in $\mathcal{F} \setminus \min(\mathcal{F})$, and so the claim $\varphi_w(J'_2) \subseteq \min(\mathcal{F} \setminus \min(\mathcal{F}))$. Therefore $|J'_2| \leq |\min(\mathcal{F} \setminus \min(\mathcal{F}))|$, and so by (9), (11) we obtain the upper bound of the statement

$$|J(\mathcal{F})| = |J_1| + |J_2 \setminus J'_2| + |J'_2| \leq 2|\min(\mathcal{F})| + |\min(\mathcal{F} \setminus \min(\mathcal{F}))|$$

□

As an immediate consequence of the previous theorem and Sperner's Theorem we have $|J(\mathcal{F})| \leq 3\binom{n}{\lfloor \frac{n}{2} \rfloor}$. This bound is not the best that can be obtained from Theorem 4. Indeed, we devote Subsection 6.1 to prove Theorem 5 showing that for an upward-closed family \mathcal{F} on a set X with $|X| = n$ we have $2|\min(\mathcal{F})| + |\min(\mathcal{F} \setminus \min(\mathcal{F}))| \leq 2\binom{n}{\lfloor \frac{n}{2} \rfloor} + \binom{n}{\lfloor \frac{n}{2} \rfloor + 1}$ and this bound is tight. Therefore we have the following corollary.

Corollary 5. *Let \mathcal{F} be a \cup -closed family of sets of 2^X with $X = \{a_1, \dots, a_n\}$, then*

$$|J(\mathcal{F})| \leq 2\binom{n}{\lfloor \frac{n}{2} \rfloor} + \binom{n}{\lfloor \frac{n}{2} \rfloor + 1}$$

In particular any family $S \subseteq 2^X$ with $|S| > 2\binom{n}{\lfloor \frac{n}{2} \rfloor} + \binom{n}{\lfloor \frac{n}{2} \rfloor + 1}$ is not \cup -independent.

A natural question that arises from this corollary is the precise upper bound of the quantity

$$J(n) = \max\{|J(\mathcal{F})| : \mathcal{F} \text{ is a } \cup\text{-closed family on a set } X \text{ with } |X| = n\}$$

Although we are not able to answer to this question we can easily give a lower bound to $J(n)$. Indeed, consider the \cup -closed family of 2^X consisting of elements whose cardinality is greater than or equal to $\lfloor \frac{n}{2} \rfloor$. The set of joint-irreducible elements consists of the subsets of cardinality exactly $\lfloor \frac{n}{2} \rfloor$, whence we can bound the function $J(n)$ as

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \leq J(n) \leq 2\binom{n}{\lfloor \frac{n}{2} \rfloor} + \binom{n}{\lfloor \frac{n}{2} \rfloor + 1}$$

6.1 An extremal problem

In this section we study the extremal problem of maximizing the quantity $2|\min(\mathcal{F})| + |\min(\mathcal{F} \setminus \min(\mathcal{F}))|$ where \mathcal{F} is an upward-closed set on the set X . We can restate this problem in the following way. Given an antichain \mathcal{A} of 2^X , we want to maximize the quantity $2|\mathcal{A}| + |\min(\mathcal{A}^\uparrow \setminus \mathcal{A})|$. Before studying this problem more in detail we give some definitions. For an integer $0 < k \leq n$ we denote by $\mathcal{A}_k = \{A \in \mathcal{A} : |A| = k\}$, in general a family of k -subsets \mathcal{B} is a collection of sets of X with cardinality k . We recall that the *shade* (see [5]) of \mathcal{A}_k is defined by

$$\nabla(\mathcal{A}_k) = \{B \in 2^X : |B| = k+1, A \subseteq B \text{ for some } A \in \mathcal{A}_k\}$$

Similarly the *shadow* of \mathcal{A}_k is defined by

$$\Delta(\mathcal{A}_k) = \{B \in 2^X : |B| = k-1, B \subseteq A \text{ for some } A \in \mathcal{A}_k\}$$

We can extend these definitions to the whole set \mathcal{A} by taking $\nabla(\mathcal{A}) = \bigcup_{k=1}^n \nabla(\mathcal{A}_k)$ and $\Delta(\mathcal{A}) = \bigcup_{k=1}^n \Delta(\mathcal{A}_k)$. Note that $\min(\mathcal{A}^\uparrow \setminus \mathcal{A}) \subseteq \nabla(\mathcal{A})$, in particular, since \mathcal{A} is an antichain, $\min(\mathcal{A}^\uparrow \setminus \mathcal{A}) = \min(\nabla(\mathcal{A}))$. Thus, it makes sense defining the *first upward level* of an antichain \mathcal{A} as the set $\overline{\nabla}\mathcal{A} = \min(\nabla(\mathcal{A}))$. The operator $\overline{\nabla}$ is also interesting because \mathcal{A}^\uparrow can be partitioned into “foils”, where the i -th foil for $i \geq 1$ is given by $\overline{\nabla}^i(\mathcal{A}) = \overline{\nabla}(\overline{\nabla}^{i-1}(\mathcal{A}))$ and $\overline{\nabla}^0(\mathcal{A}) = \mathcal{A}$. We state some useful properties whose proofs are left to the reader.

Lemma 12. *Let \mathcal{A}, \mathcal{B} be two antichains, then:*

1. $\nabla(\mathcal{A} \cup \mathcal{B}) = \nabla(\mathcal{A}) \cup \nabla(\mathcal{B})$, $\Delta(\mathcal{A} \cup \mathcal{B}) = \Delta(\mathcal{A}) \cup \Delta(\mathcal{B})$, $\mathcal{A} \subseteq \nabla(\Delta(\mathcal{A}))$, $\mathcal{A} \subseteq \Delta(\nabla(\mathcal{A}))$.
2. Assume $\mathcal{A} \subseteq \mathcal{B}$, then for any $g \in \nabla(\mathcal{A})$ there is a $g' \in \overline{\nabla}(\mathcal{B})$ such that $g' \subseteq g$.
3. $\overline{\nabla}(\mathcal{A}) \subseteq \nabla(\mathcal{A})$, moreover $g \in \nabla(\mathcal{A}) \setminus \overline{\nabla}(\mathcal{A})$ iff there is $g' \in \nabla(\mathcal{A})$ such that $g' \subsetneq g$.
4. If $\mathcal{A} \cup \mathcal{B}$ is an antichain and for all $g \in \overline{\nabla}(\mathcal{A})$ there is no $g' \in \overline{\nabla}(\mathcal{B})$ such that $g' \subsetneq g$, then $\overline{\nabla}(\mathcal{A}) \subseteq \overline{\nabla}(\mathcal{A} \cup \mathcal{B})$.
5. Assume $\mathcal{A} \subseteq \mathcal{B}$, if for any $g \in \nabla(\mathcal{A})$ there is a $g' \in \nabla(\mathcal{B} \setminus \mathcal{A})$ such that $g' \subseteq g$, then $\overline{\nabla}(\mathcal{B} \setminus \mathcal{A}) = \overline{\nabla}(\mathcal{B})$.

We devote the rest of the paper to the proof of following theorem.

Theorem 5. *Let \mathcal{A} be an antichain of 2^X with $|X| = n$, then*

$$2|\mathcal{A}| + |\overline{\nabla}(\mathcal{A})| \leq 2 \binom{n}{\lfloor \frac{n}{2} \rfloor} + \binom{n}{\lfloor \frac{n}{2} \rfloor + 1}$$

and this bound is tight.

Note that if n is odd the theorem can be easily proved. Indeed, both $|\mathcal{A}|$ and $|\overline{\nabla}(\mathcal{A})|$ are antichains, hence by Sperner's theorem we get $2|\mathcal{A}| + |\overline{\nabla}(\mathcal{A})| \leq 3 \binom{n}{\lfloor \frac{n}{2} \rfloor} = 2 \binom{n}{\lfloor \frac{n}{2} \rfloor} + \binom{n}{\lfloor \frac{n}{2} \rfloor + 1}$ since $\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lfloor \frac{n}{2} \rfloor + 1}$. It is not difficult to check that this bound is attained when \mathcal{A} consists of all the $\frac{n-1}{2}$ -subsets of X . Therefore, in the sequel we can assume that n is even. We prove the theorem using an augmentation argument. More precisely, we define two maps α^+, α^- , called respectively the *upward-augmenting*, *lower-augmenting* map, with the property of transforming \mathcal{A} into the antichains $\alpha^+(\mathcal{A}), \alpha^-(\mathcal{A})$ with

$$\begin{aligned} 2|\mathcal{A}| + |\overline{\nabla}(\mathcal{A})| &\leq 2|\alpha^+(\mathcal{A})| + |\overline{\nabla}(\alpha^+(\mathcal{A}))| \\ 2|\mathcal{A}| + |\overline{\nabla}(\mathcal{A})| &\leq 2|\alpha^-(\mathcal{A})| + |\overline{\nabla}(\alpha^-(\mathcal{A}))| \end{aligned}$$

then we repetitively apply these operators to obtain an antichain formed by k -subsets of X with $k = \frac{n}{2}, \frac{n}{2} - 1$. However we define these maps only for particular classes of antichains that we are going to introduce, first we need some preliminary definitions. Given a family $\mathcal{B} \subseteq 2^X$ we denote the maximum (minimum) of the lengths of the elements of \mathcal{B} by $\|\mathcal{B}\|_M$ ($\|\mathcal{B}\|_m$), and we put $Max(\mathcal{B}) = \{B \in \mathcal{B} : |B| = \|\mathcal{B}\|_M\}$, $Min(\mathcal{B}) = \{B \in \mathcal{B} : |B| = \|\mathcal{B}\|_m\}$. The following lemma shows that we can restrict our attention to a particular class of antichains.

Lemma 13. *Let \mathcal{A}' be an antichain in 2^X , then*

1) *for any $h \in Min(\mathcal{A}')$ and $a \in X \setminus h$ we have $h \cup \{a\} \in \overline{\nabla}(\mathcal{A}')$.*

Moreover there is an antichain $\mathcal{A} \supseteq \mathcal{A}'$ such that $\|\mathcal{A}\|_M = \|\mathcal{A}'\|_M$, $\|\mathcal{A}\|_m = \|\mathcal{A}'\|_m$, $2|\mathcal{A}'| + |\overline{\nabla}(\mathcal{A}')| \leq 2|\mathcal{A}| + |\overline{\nabla}(\mathcal{A})|$ and with the following property:

2) *let $k = \|\overline{\nabla}(\mathcal{A})\|_M$, then either $\cup_{i \geq k} \mathcal{A}_i \neq \emptyset$ or for any $h \in Max(\overline{\nabla}(\mathcal{A}))$ and $a \in h$ we have $h \setminus \{a\} \in \mathcal{A}$.*

Proof. Let us prove Condition 1). To obtain a contradiction suppose that there is $h \in Min(\mathcal{A}')$ and $a \in X \setminus h$ such that $h \cup \{a\} \notin \overline{\nabla}(\mathcal{A}')$. Thus $h \cup \{a\} \in \nabla(\mathcal{A}') \setminus \overline{\nabla}(\mathcal{A}')$, and so by property 3 of Lemma 12, there is a

$g \in \nabla(\mathcal{A}')$ and $h' \in \mathcal{A}'$ with $h' \subsetneq g \subsetneq h \cup \{a\}$, whence $|h'| < |g| < |h| + 1$. Thus $|h'| < |h|$ which contradicts the minimality of $|h|$.

The second statement is proved if we show that given an antichain \mathcal{B}' with $k = \|\overline{\nabla}(\mathcal{B}')\|_M$ either $\cup_{i \geq k} \mathcal{B}'_i \neq \emptyset$ or if there is an $h \in \text{Max}(\overline{\nabla}(\mathcal{B}'))$ and $a \in h$ such that $h \setminus \{a\} \notin \mathcal{B}'$, then the family $\mathcal{B} = \mathcal{B}' \cup \{h \setminus \{a\}\}$ is an antichain with $2|\mathcal{B}'| + |\overline{\nabla}(\mathcal{B}')| \leq 2|\mathcal{B}| + |\overline{\nabla}(\mathcal{B})|$. Indeed starting from \mathcal{A}' by repetitively adding elements for which condition 2) does not hold, we eventually end with an antichain \mathcal{A} satisfying property 2). Suppose that $\cup_{i \geq k} \mathcal{B}'_i = \emptyset$, otherwise we have done. It is easily seen that $\|\mathcal{B}\|_M = \|\mathcal{B}'\|_M$, $\|\mathcal{B}\|_m = \|\mathcal{B}'\|_m$. We now prove that \mathcal{B} is an antichain. Note first that h can not be a singleton, thus to reach a contradiction suppose that \mathcal{B} is not an antichain. Since \mathcal{B}' is an antichain, there is a $g \in \mathcal{B}'$ such that either $g \subsetneq h \setminus \{a\}$ or $h \setminus \{a\} \subsetneq g$. Suppose that $g \subsetneq h \setminus \{a\}$, hence there is a $h' \in \overline{\nabla}(\mathcal{B}')$ such that $h' \subseteq g \cup \{a\} \subsetneq h$ which contradicts the fact that $\overline{\nabla}(\mathcal{B}')$ is an antichain. On the other hand suppose that $h \setminus \{a\} \subsetneq g$. Thus, $|g| \geq |h| = k$ which implies $g \in \cup_{i \geq k} \mathcal{B}'_i = \emptyset$, a contradiction. Therefore $\mathcal{B} = \mathcal{B}' \cup \{h \setminus \{a\}\}$ is an antichain. We now prove that $\overline{\nabla}(\mathcal{B}') \subseteq \overline{\nabla}(\mathcal{B})$ from which, with $\mathcal{B} = \mathcal{B}' \cup \{h \setminus \{a\}\}$, implies our claim $2|\mathcal{B}'| + |\overline{\nabla}(\mathcal{B}')| \leq 2|\mathcal{B}| + |\overline{\nabla}(\mathcal{B})|$. Suppose, contrary to our claim, that there is a $t \in \overline{\nabla}(\mathcal{B}') \setminus \overline{\nabla}(\mathcal{B}) \neq \emptyset$. It is straightforward to check that there is a $t' \in \overline{\nabla}(\mathcal{B})$ with $t' \subsetneq t$. It follows easily that $h \setminus \{a\} \subsetneq t' \subsetneq t$ (otherwise we would have the contradiction $t' \in \overline{\nabla}(\mathcal{B}')$). Thus we have $|t| > |t'| \geq |h| = k$, against $\|\overline{\nabla}(\mathcal{B}')\|_M = k$. \square

An antichain \mathcal{A} satisfying the two properties in Lemma 13 is called *augmentable*. We now define the lower-augmenting, upward-augmenting map on the set of augmentable antichains over X . Given an augmentable antichain \mathcal{A} with $k = \|\overline{\nabla}(\mathcal{A})\|_M$, $k' = \|\mathcal{A}\|_M$, $s = \|\mathcal{A}\|_m$, the lower-augmenting map is defined by

$$\alpha^-(\mathcal{A}) = \begin{cases} (\mathcal{A} \setminus \mathcal{A}_{k'}) \cup \Delta(\mathcal{A}_{k'}), & \text{if } k' \geq k, k' \geq \frac{n}{2} + 1 \\ (\mathcal{A} \setminus \mathcal{A}_{k-1}) \cup \Delta(\mathcal{A}_{k-1}), & \text{if } k' < k, k > \frac{n}{2} + 1 \\ \mathcal{A} & \text{otherwise.} \end{cases}$$

and the upward-augmenting map by

$$\alpha^+(\mathcal{A}) = \begin{cases} (\mathcal{A} \setminus \mathcal{A}_s) \cup \nabla(\mathcal{A}_s), & \text{if } s < \frac{n}{2} - 1 \\ \mathcal{A} & \text{otherwise.} \end{cases}$$

The following lemma shows that $\alpha^+(\mathcal{A}), \alpha^-(\mathcal{A})$ are antichains.

Lemma 14. *Let \mathcal{A} be an antichain and let $M = \|\mathcal{A}\|_M$, $m = \|\mathcal{A}\|_m$, then*

$$\mathcal{A} \setminus \mathcal{A}_m \cup \nabla(\mathcal{A}_m), \mathcal{A} \setminus \mathcal{A}_M \cup \Delta(\mathcal{A}_M)$$

are antichains with $\mathcal{A} \setminus \mathcal{A}_m \cap \nabla(\mathcal{A}_m) = \emptyset$, $\mathcal{A} \setminus \mathcal{A}_M \cap \Delta(\mathcal{A}_M) = \emptyset$.

Proof. Suppose, contrary to our claim, that there is $g \in \mathcal{A} \setminus \mathcal{A}_m \cap \nabla(\mathcal{A}_m) \neq \emptyset$. Thus $g \in \nabla(\mathcal{A}_m)$ implies that there is a $g' \in \mathcal{A}_m$ such that $g' \subsetneq g$ which contradicts the fact that \mathcal{A} is an antichain. Similarly, $\mathcal{A}_M \cap \Delta(\mathcal{A}_M) \neq \emptyset$ contradicts the fact that \mathcal{A} is an antichain. Let us prove that $\mathcal{A} \setminus \mathcal{A}_m \cup \nabla(\mathcal{A}_m)$ is an antichain. Since the two terms of the union are disjoint antichains, to reach a contradiction, we can suppose that there is a $g \in \mathcal{A} \setminus \mathcal{A}_m$ and $g' \in \nabla(\mathcal{A}_m)$ such that either $g \subsetneq g'$ or $g' \subsetneq g$. Since m is the minimum of the length of the elements of \mathcal{A} , then $g \in \mathcal{A} \setminus \mathcal{A}_m$ implies $|g| \geq m+1$, while $g' \in \nabla(\mathcal{A}_m)$ implies $|g'| = m+1$. Thus only $g' \subsetneq g$ can occur. However $g' \in \nabla(\mathcal{A}_m)$ implies $g'' \subsetneq g'$, for some $g'' \in \mathcal{A}_m$. Hence $g'' \subsetneq g' \subsetneq g$ which contradicts the fact that \mathcal{A} is an antichain. Hence $\mathcal{A} \setminus \mathcal{A}_m \cup \nabla(\mathcal{A}_m)$ is an antichain. Let us prove that $\mathcal{A} \setminus \mathcal{A}_M \cup \Delta(\mathcal{A}_M)$ is also an antichain. Suppose, contrary to our claim, that $\mathcal{A} \setminus \mathcal{A}_M \cup \Delta(\mathcal{A}_M)$ is not an antichain. Similarly to the above situation, we can assume that only $g \subsetneq g'$ for $g \in \mathcal{A} \setminus \mathcal{A}_M$ and $g' \in \Delta(\mathcal{A}_M)$ can occur. However, $g' \in \Delta(\mathcal{A}_M)$ implies that there is a $g'' \in \mathcal{A}_M$ with $g' \subsetneq g''$, hence $g \subsetneq g' \subsetneq g''$ contradicts the fact that \mathcal{A} is an antichain. Therefore $\mathcal{A} \setminus \mathcal{A}_M \cup \Delta(\mathcal{A}_M)$ is an antichain and this concludes the proof of the lemma. \square

Lemma 15. *Let \mathcal{A} be an augmentable antichain, then $\alpha^+(\mathcal{A})$ is antichain with $\|\alpha^+(\mathcal{A})\|_m > \|\mathcal{A}\|_m$, if $\|\mathcal{A}\|_m < \|\mathcal{A}\|_M$ then $\|\alpha^+(\mathcal{A})\|_M = \|\mathcal{A}\|_M$, moreover:*

$$2|\mathcal{A}| + |\overline{\nabla}(\mathcal{A})| \leq 2|\alpha^+(\mathcal{A})| + |\overline{\nabla}(\alpha^+(\mathcal{A}))|$$

Proof. Let $s = \|\mathcal{A}\|_m$, it is evident that α^+ substitutes $\text{Min}(\mathcal{A})$ with $\nabla(\text{Min}(\mathcal{A}))$. Thus $\|\alpha^+(\mathcal{A})\|_m > \|\mathcal{A}\|_m$. Moreover if $s < \|\mathcal{A}\|_M$ then, since we just add elements of cardinality $s+1$, it is also immediate that $\|\alpha^+(\mathcal{A})\|_M = \|\mathcal{A}\|_M$.

By Lemma 14 $\alpha^+(\mathcal{A})$ is an antichain with:

$$\mathcal{A} \setminus \mathcal{A}_s \cap \nabla(\mathcal{A}_s) = \emptyset \tag{12}$$

Let us prove the inequality $2|\mathcal{A}| + |\overline{\nabla}(\mathcal{A})| \leq 2|\alpha^+(\mathcal{A})| + |\overline{\nabla}(\alpha^+(\mathcal{A}))|$. We first claim that

$$\overline{\nabla}(\alpha^+(\mathcal{A})) \supseteq \overline{\nabla}(\mathcal{A}) \setminus \nabla(\mathcal{A}_s) \cup \nabla(\nabla(\mathcal{A}_s)) \tag{13}$$

where $\nabla(\mathcal{A}_s) \subseteq \overline{\nabla}(\mathcal{A})$ and

$$\overline{\nabla}(\mathcal{A}) \setminus \nabla(\mathcal{A}_s) \cap \nabla(\nabla(\mathcal{A}_s)) = \emptyset \tag{14}$$

By property 1) of an augmentable chain \mathcal{A} we have $\nabla(\mathcal{A}_s) \subseteq \overline{\nabla}(\mathcal{A})$. Let us prove (14). Suppose that (14) do not hold and let $h \in \overline{\nabla}(\mathcal{A}) \setminus \nabla(\mathcal{A}_s) \cap \nabla(\nabla(\mathcal{A}_s))$. Thus $h = g \cup \{a, b\}$ for some $g \in \mathcal{A}_s$ and $a, b \notin g$, since $g' = g \cup \{a\} \in \overline{\nabla}(\mathcal{A})$ we have $g' \subsetneq h$ for $g', h \in \overline{\nabla}(\mathcal{A})$, a contradiction. Let us prove (13). We split the proof of (13) by showing first $\overline{\nabla}(\mathcal{A}) \setminus \nabla(\mathcal{A}_s) \subseteq \overline{\nabla}(\mathcal{A} \setminus \mathcal{A}_s \cup \nabla(\mathcal{A}_s))$ and then $\nabla(\nabla(\mathcal{A}_s)) \subseteq \overline{\nabla}(\mathcal{A} \setminus \mathcal{A}_s \cup \nabla(\mathcal{A}_s))$.

- Case $\overline{\nabla}(\mathcal{A}) \setminus \nabla(\mathcal{A}_s) \subseteq \overline{\nabla}(\mathcal{A} \setminus \mathcal{A}_s \cup \nabla(\mathcal{A}_s))$. Since $\overline{\nabla}(\mathcal{A})_{s+1} \subseteq \nabla(\mathcal{A}_s) \subseteq \overline{\nabla}(\mathcal{A})_{s+1}$, then $\overline{\nabla}(\mathcal{A})_{s+1} = \nabla(\mathcal{A}_s)$. Thus $\overline{\nabla}(\mathcal{A}) \setminus \nabla(\mathcal{A}_s) \subseteq \nabla(\mathcal{A} \setminus \mathcal{A}_s)$, and so, by property 1) of Lemma 12, we get $\overline{\nabla}(\mathcal{A}) \setminus \nabla(\mathcal{A}_s) \subseteq \nabla(\mathcal{A} \setminus \mathcal{A}_s \cup \nabla(\mathcal{A}_s))$. Suppose, contrary to our claim, that there is a $g \in \overline{\nabla}(\mathcal{A}) \setminus \nabla(\mathcal{A}_s)$ such that $g \in \nabla(\mathcal{A} \setminus \mathcal{A}_s \cup \nabla(\mathcal{A}_s)) \setminus \overline{\nabla}(\mathcal{A} \setminus \mathcal{A}_s \cup \nabla(\mathcal{A}_s))$. Therefore, by properties 3), 1) of Lemma 12 there is a $g' \in \nabla(\mathcal{A} \setminus \mathcal{A}_s \cup \nabla(\mathcal{A}_s)) = \nabla(\mathcal{A} \setminus \mathcal{A}_s) \cup \nabla(\nabla(\mathcal{A}_s))$ such that $g' \subsetneq g$. We consider two cases, either $g' \in \nabla(\mathcal{A} \setminus \mathcal{A}_s)$ or $g' \in \nabla(\nabla(\mathcal{A}_s))$. Suppose that $g' \in \nabla(\mathcal{A} \setminus \mathcal{A}_s)$, then by property 2) of Lemma 12, there is a $g'' \in \overline{\nabla}(\mathcal{A})$ such that $g'' \subseteq g' \subsetneq g \in \overline{\nabla}(\mathcal{A})$ which contradicts the fact that $\overline{\nabla}(\mathcal{A})$ is an antichain. On the other hand, suppose that $g' \in \nabla(\nabla(\mathcal{A}_s))$. Hence there is a $g'' \in \nabla(\mathcal{A}_s) \subseteq \overline{\nabla}(\mathcal{A})$ such that $g'' \subsetneq g' \subsetneq g \in \overline{\nabla}(\mathcal{A})$ which again contradicts the fact that $\overline{\nabla}(\mathcal{A})$ is an antichain. Hence we conclude $\overline{\nabla}(\mathcal{A}) \setminus \nabla(\mathcal{A}_s) \subseteq \overline{\nabla}(\mathcal{A} \setminus \mathcal{A}_s \cup \nabla(\mathcal{A}_s))$.
- Case $\nabla(\nabla(\mathcal{A}_s)) \subseteq \overline{\nabla}(\mathcal{A} \setminus \mathcal{A}_s \cup \nabla(\mathcal{A}_s))$. It is evident by property 2) of Lemma 12 that $\nabla(\nabla(\mathcal{A}_s)) \subseteq \nabla(\mathcal{A} \setminus \mathcal{A}_s \cup \nabla(\mathcal{A}_s))$. Suppose, contrary to our claim, that there is a $g \in \nabla(\nabla(\mathcal{A}_s))$ such that $g \in \nabla(\mathcal{A} \setminus \mathcal{A}_s \cup \nabla(\mathcal{A}_s)) \setminus \overline{\nabla}(\mathcal{A} \setminus \mathcal{A}_s \cup \nabla(\mathcal{A}_s))$. Therefore, by properties 3), 1) of Lemma 12 there is a $g' \in \nabla(\mathcal{A} \setminus \mathcal{A}_s \cup \nabla(\mathcal{A}_s)) = \nabla(\mathcal{A} \setminus \mathcal{A}_s) \cup \nabla(\nabla(\mathcal{A}_s))$ such that $g' \subsetneq g$. Also in this case we consider the two cases either $g' \in \nabla(\mathcal{A} \setminus \mathcal{A}_s)$ or $g' \in \nabla(\nabla(\mathcal{A}_s))$. Suppose that $g' \in \nabla(\mathcal{A} \setminus \mathcal{A}_s)$. Since $s = \|\mathcal{A}\|_m$, then $|g'| \geq s + 2$, while $g \in \nabla(\nabla(\mathcal{A}_s))$ implies $|g| = s + 2$ which contradicts $g' \subsetneq g$. In the other case, if $g' \in \nabla(\nabla(\mathcal{A}_s))$, then $g \in \nabla(\nabla(\mathcal{A}_s))$ and $g' \subsetneq g$ contradict the fact that $\nabla(\nabla(\mathcal{A}_s))$ is an antichain. Hence $\nabla(\nabla(\mathcal{A}_s)) \subseteq \overline{\nabla}(\mathcal{A} \setminus \mathcal{A}_s \cup \nabla(\mathcal{A}_s))$ and this completes the proof of (13).

Let us complete the proof of the lemma showing the inequality in the statement. Since $s < \frac{n}{2} - 1$, then by [5, Corollary 2.1.2], $|\nabla(\mathcal{A}_s)| - |\mathcal{A}_s| \geq 0$ and $|\nabla(\nabla(\mathcal{A}_s))| - |\nabla(\mathcal{A}_s)| \geq 0$. By (12), (13), (14) we have $2|\alpha^+(\mathcal{A})| + |\overline{\nabla}(\alpha^+(\mathcal{A}))| \geq 2|\mathcal{A}| - 2|\mathcal{A}_s| + 2|\nabla(\mathcal{A}_s)| + |\overline{\nabla}(\mathcal{A}) \setminus \nabla(\mathcal{A}_s)| + |\nabla(\nabla(\mathcal{A}_s))|$. Furthermore, using $\nabla(\mathcal{A}_s) \subseteq \overline{\nabla}(\mathcal{A})$, $|\nabla(\mathcal{A}_s)| - |\mathcal{A}_s| \geq 0$ and $|\nabla(\nabla(\mathcal{A}_s))| -$

$|\nabla(\mathcal{A}_s)| \geq 0$ we get $2|\alpha^+(\mathcal{A})| + |\overline{\nabla}(\alpha^+(\mathcal{A}))| \geq 2|\mathcal{A}| + |\overline{\nabla}(\mathcal{A})|$. \square

Lemma 16. *Let \mathcal{A} be an augmentable antichain, then $\alpha^-(\mathcal{A})$ is an antichain with $\|\alpha^-(\mathcal{A})\|_M < \|\mathcal{A}\|_M$, if $\|\mathcal{A}\|_m < \|\mathcal{A}\|_M$ then $\|\alpha^-(\mathcal{A})\|_m = \|\mathcal{A}\|_m$, moreover:*

$$2|\mathcal{A}| + |\overline{\nabla}(\mathcal{A})| \leq 2|\alpha^-(\mathcal{A})| + |\overline{\nabla}(\alpha^-(\mathcal{A}))|$$

Proof. Let $k = \|\overline{\nabla}(\mathcal{A})\|_M, k' = \|\mathcal{A}\|_M$. For this operator we need to consider two cases: either $k' \geq k$ and $k' \geq \frac{n}{2} + 1$, or $k' < k$ and $k > \frac{n}{2} + 1$. Note that in the case $k' < k$, since \mathcal{A} is augmentable, then by property 2) of Lemma 13 we have $k' = k - 1$. In both cases the map α^- substitutes $\text{Max}(\mathcal{A})$ with $\Delta(\text{Max}(\mathcal{A}))$, thus $\|\alpha^-(\mathcal{A})\|_M < \|\mathcal{A}\|_M$ holds and if $\|\mathcal{A}\|_m < \|\mathcal{A}\|_M$ then it is also obvious that $\|\alpha^-(\mathcal{A})\|_m = \|\mathcal{A}\|_m$.

Consider now the case $k' \geq k$. By Lemma 14, $\alpha^-(\mathcal{A}) = (\mathcal{A} \setminus \mathcal{A}_{k'}) \cup \Delta(\mathcal{A}_{k'})$ is an antichain with:

$$(\mathcal{A} \setminus \mathcal{A}_{k'}) \cap \Delta(\mathcal{A}_{k'}) = \emptyset \quad (15)$$

We now prove the inequality of the statement. We claim

$$\overline{\nabla}(\alpha^-(\mathcal{A})) \supseteq \overline{\nabla}(\mathcal{A}) \quad (16)$$

We first prove that $\overline{\nabla}(\mathcal{A} \setminus \mathcal{A}_{k'}) = \overline{\nabla}(\mathcal{A})$. Since $k = \|\overline{\nabla}(\mathcal{A})\|_M$ and $k' \geq k$, then any element in $\nabla(\mathcal{A}_{k'})$ contains some element in $\nabla(\mathcal{A} \setminus \mathcal{A}_{k'})$. Thus by property 5) of Lemma 12 we have the claim $\overline{\nabla}(\mathcal{A} \setminus \mathcal{A}_{k'}) = \overline{\nabla}(\mathcal{A})$. If we show that for any $g \in \overline{\nabla}(\mathcal{A})$ there is no $g' \in \overline{\nabla}(\Delta(\mathcal{A}_{k'}))$ such that $g' \subsetneq g$, then the inclusion (16) follows from property 4) of Lemma 12 and $\overline{\nabla}(\mathcal{A} \setminus \mathcal{A}_{k'}) = \overline{\nabla}(\mathcal{A})$. Indeed suppose, contrary to our claim, that there are $g \in \overline{\nabla}(\mathcal{A}), g' \in \overline{\nabla}(\Delta(\mathcal{A}_{k'}))$ such that $g' \subsetneq g$. Since $\overline{\nabla}(\mathcal{A})$ is formed by elements of cardinality less or equal to k and $\overline{\nabla}(\Delta(\mathcal{A}_{k'}))$ of elements whose cardinality is $k' \geq k$ we have the contradiction $k \geq |g| > |g'| \geq k$ and this concludes the proof of (16). We now prove the inequality in the statement of the lemma. By (15) and inclusion (16) we have $2|\alpha^-(\mathcal{A})| + |\overline{\nabla}(\alpha^-(\mathcal{A}))| = 2|\mathcal{A}| - 2|\mathcal{A}_{k'}| + 2|\Delta(\mathcal{A}_{k'})| + |\overline{\nabla}(\alpha^-(\mathcal{A}))| \geq 2|\mathcal{A}| + 2(|\Delta(\mathcal{A}_{k'})| - |\mathcal{A}_{k'}|) + |\overline{\nabla}(\mathcal{A})|$. Since $k' \geq \frac{n}{2} + 1$, then by [5, Corollary 2.1.2] we have $|\Delta(\mathcal{A}_{k'})| - |\mathcal{A}_{k'}| \geq 0$, whence $2|\alpha^-(\mathcal{A})| + |\overline{\nabla}(\alpha^-(\mathcal{A}))| \geq 2|\mathcal{A}| + |\overline{\nabla}(\mathcal{A})|$.

Consider now the other case $k' < k$ and $k > \frac{n}{2} + 1$. Therefore, by Lemma 14, $\alpha^-(\mathcal{A}) = (\mathcal{A} \setminus \mathcal{A}_{k-1}) \cup \Delta(\mathcal{A}_{k-1})$ is an antichain with:

$$(\mathcal{A} \setminus \mathcal{A}_{k-1}) \cap \Delta(\mathcal{A}_{k-1}) = \emptyset \quad (17)$$

We now prove the inequality of the statement. We first prove the following inclusion:

$$\overline{\nabla}((\mathcal{A} \setminus \mathcal{A}_{k-1}) \cup \Delta(\mathcal{A}_{k-1})) \supseteq \overline{\nabla}(\mathcal{A}) \setminus \overline{\nabla}(\mathcal{A})_k \cup \Delta(\overline{\nabla}(\mathcal{A})_k) \quad (18)$$

with $\overline{\nabla}(\mathcal{A}) \setminus \overline{\nabla}(\mathcal{A})_k \cap \Delta(\overline{\nabla}(\mathcal{A})_k) = \emptyset$. Let us prove first this last property. Suppose on the contrary that there is an $h \in \overline{\nabla}(\mathcal{A}) \setminus \overline{\nabla}(\mathcal{A})_k \cap \Delta(\overline{\nabla}(\mathcal{A})_k) \neq \emptyset$. Since \mathcal{A} is augmentable and $k' < k$, then by property 2) of Lemma 13 we have

$$\Delta(\overline{\nabla}(\mathcal{A})_k) \subseteq \mathcal{A}_{k-1} \quad (19)$$

Therefore we have $h \in \mathcal{A}_{k-1} \cap \overline{\nabla}(\mathcal{A}) \subseteq \mathcal{A} \cap \overline{\nabla}(\mathcal{A}) = \emptyset$, a contradiction. Hence the two terms in the right part of the inclusion (18) are disjoint. We divide the proof of (18) into two cases. We first prove $\overline{\nabla}(\mathcal{A}) \setminus \overline{\nabla}(\mathcal{A})_k \subseteq \overline{\nabla}((\mathcal{A} \setminus \mathcal{A}_{k-1}) \cup \Delta(\mathcal{A}_{k-1}))$ and then $\Delta(\overline{\nabla}(\mathcal{A})_k) \subseteq \overline{\nabla}((\mathcal{A} \setminus \mathcal{A}_{k-1}) \cup \Delta(\mathcal{A}_{k-1}))$.

- Case $\overline{\nabla}(\mathcal{A}) \setminus \overline{\nabla}(\mathcal{A})_k \subseteq \overline{\nabla}((\mathcal{A} \setminus \mathcal{A}_{k-1}) \cup \Delta(\mathcal{A}_{k-1}))$. It is evident that $\overline{\nabla}(\mathcal{A}) \setminus \overline{\nabla}(\mathcal{A})_k \subseteq \overline{\nabla}(\mathcal{A} \setminus \mathcal{A}_{k-1})$, thus it is sufficient to show $\overline{\nabla}(\mathcal{A} \setminus \mathcal{A}_{k-1}) \subseteq \overline{\nabla}((\mathcal{A} \setminus \mathcal{A}_{k-1}) \cup \Delta(\mathcal{A}_{k-1}))$ and to prove this inclusion we use property 4) of Lemma 12. Indeed consider $g \in \overline{\nabla}(\mathcal{A} \setminus \mathcal{A}_{k-1})$ and $g' \in \overline{\nabla}(\Delta(\mathcal{A}_{k-1}))$, then $|g| \leq k-1$, $|g'| = k-1$. Therefore the inclusion $g' \subsetneq g$ can not occur and so $\overline{\nabla}(\mathcal{A}) \setminus \overline{\nabla}(\mathcal{A})_k \subseteq \overline{\nabla}(\mathcal{A} \setminus \mathcal{A}_{k-1}) \subseteq \overline{\nabla}((\mathcal{A} \setminus \mathcal{A}_{k-1}) \cup \Delta(\mathcal{A}_{k-1}))$.
- Case $\Delta(\overline{\nabla}(\mathcal{A})_k) \subseteq \overline{\nabla}((\mathcal{A} \setminus \mathcal{A}_{k-1}) \cup \Delta(\mathcal{A}_{k-1}))$. Using (19) and property 1) of Lemma 12 we have

$$\Delta(\overline{\nabla}(\mathcal{A})_k) \subseteq \mathcal{A}_{k-1} \subseteq \nabla(\Delta(\mathcal{A}_{k-1})) \subseteq \nabla((\mathcal{A} \setminus \mathcal{A}_{k-1}) \cup \Delta(\mathcal{A}_{k-1}))$$

To reach a contradiction suppose that there is a $g \in \Delta(\overline{\nabla}(\mathcal{A})_k)$ such that $g \in \nabla((\mathcal{A} \setminus \mathcal{A}_{k-1}) \cup \Delta(\mathcal{A}_{k-1})) \setminus \overline{\nabla}((\mathcal{A} \setminus \mathcal{A}_{k-1}) \cup \Delta(\mathcal{A}_{k-1}))$. By property 3) of Lemma 12 there is a $g' \in \nabla((\mathcal{A} \setminus \mathcal{A}_{k-1}) \cup \Delta(\mathcal{A}_{k-1})) = \nabla(\mathcal{A} \setminus \mathcal{A}_{k-1}) \cup \nabla(\Delta(\mathcal{A}_{k-1}))$ with $g' \subsetneq g$. We consider two cases, either $g' \in \nabla(\mathcal{A} \setminus \mathcal{A}_{k-1})$ or $g' \in \nabla(\Delta(\mathcal{A}_{k-1}))$. If $g' \in \nabla(\mathcal{A} \setminus \mathcal{A}_{k-1})$, then there is a $g'' \in \mathcal{A} \setminus \mathcal{A}_{k-1}$ such that $g'' \subsetneq g' \subsetneq g$, a contradiction since $g'' \in \mathcal{A}$, $g \in \Delta(\overline{\nabla}(\mathcal{A})_k) \subseteq \mathcal{A}_{k-1} \subseteq \mathcal{A}$ and \mathcal{A} is an antichain. On the other hand, if $g' \in \nabla(\Delta(\mathcal{A}_{k-1}))$ then $|g'| = k-1$, moreover, since $g \in \Delta(\overline{\nabla}(\mathcal{A})_k) \subseteq \mathcal{A}_{k-1}$, then $|g| = k-1$ which contradicts $g' \subsetneq g$ and this concludes the proof of inclusion (18).

We can now conclude the proof of the lemma showing the inequality in the statement. By (17) and (18) we have

$$\begin{aligned} 2|\alpha^-(\mathcal{A})| + |\overline{\nabla}(\alpha^-(\mathcal{A}))| &= 2|\mathcal{A}| + 2(|\Delta(\mathcal{A}_{k-1})| - |\mathcal{A}_{k-1}|) + |\overline{\nabla}(\alpha^-(\mathcal{A}))| \\ &\geq 2|\mathcal{A}| + |\overline{\nabla}(\mathcal{A})| + 2(|\Delta(\mathcal{A}_{k-1})| - |\mathcal{A}_{k-1}|) + \\ &\quad + (|\Delta(\overline{\nabla}(\mathcal{A})_k)| - |\overline{\nabla}(\mathcal{A})_k|) \end{aligned}$$

Since $k > \frac{n}{2} + 1$ we have by [5, Corollary 2.1.2]

$$|\Delta(\mathcal{A}_{k-1})| - |\mathcal{A}_{k-1}| \geq 0, |\Delta(\overline{\nabla}(\mathcal{A})_k)| - |\overline{\nabla}(\mathcal{A})_k| \geq 0$$

from which it follows $2|\alpha^-(\mathcal{A})| + |\overline{\nabla}(\alpha^-(\mathcal{A}))| \geq 2|\mathcal{A}| + |\overline{\nabla}(\mathcal{A})|$ and this concludes the proof of the lemma. \square

We are now in position to prove Theorem 5.

Proof of Theorem 5. The bound is clearly attained when the antichain consists of all the $\frac{n}{2}$ -subsets of X . Let us now prove the bound. Starting from $\mathcal{A}_0 = \mathcal{A}$ by Lemma 13 we suppose without loss of generality that \mathcal{A}_0 is augmentable, then applying for instance the upward-augmenting map we obtain a new antichain \mathcal{A}_1 for which, by Lemmas 15, $2|\mathcal{A}_0| + |\overline{\nabla}(\mathcal{A}_0)| \leq 2|\mathcal{A}_1| + |\overline{\nabla}(\mathcal{A}_1)|$ and $\|\mathcal{A}_1\|_m > \|\mathcal{A}_0\|_m$. Furthermore by Lemma 13 we can suppose that \mathcal{A}_1 is also augmentable. In this way, by a repeated application of Lemmas 13, 15, 16 we can find a sequence of augmentable antichains \mathcal{A}_i such that $2|\mathcal{A}_{i-1}| + |\overline{\nabla}(\mathcal{A}_{i-1})| \leq 2|\mathcal{A}_i| + |\overline{\nabla}(\mathcal{A}_i)|$ and either $\|\mathcal{A}_i\|_m > \|\mathcal{A}_{i-1}\|_m$ and $\|\mathcal{A}_i\|_M = \|\mathcal{A}_{i-1}\|_M$, or $\|\mathcal{A}_i\|_M < \|\mathcal{A}_{i-1}\|_M$ and $\|\mathcal{A}_i\|_m = \|\mathcal{A}_{i-1}\|_m$. This process stops when it is reached an augmentable antichain \mathcal{A}_j with $\frac{n}{2} \geq \|\mathcal{A}_j\|_M \geq \|\mathcal{A}_j\|_m \geq \frac{n}{2} - 1$. If $\|\mathcal{A}_j\|_M = \|\mathcal{A}_j\|_m$, \mathcal{A}_j consists of either $\frac{n}{2}$ -subsets or $\frac{n}{2} - 1$ -subsets and the statement of the theorem clearly holds. Thus we can assume $\|\mathcal{A}_j\|_M > \|\mathcal{A}_j\|_m$ and let $\mathcal{B}_1 = \text{Min}(\mathcal{A}_j)$, $\mathcal{B}_2 = \text{Max}(\mathcal{A}_j)$. Since \mathcal{A}_j is augmentable, by property 1) of by Lemma 13, $\nabla(\mathcal{B}_1) \subseteq \overline{\nabla}\mathcal{A}_j$. Thus, putting $\mathcal{C}_1 = \nabla(\mathcal{B}_1)$, we can decompose $\overline{\nabla}\mathcal{A}_j = \mathcal{C}_1 \cup \mathcal{C}_2$ where $\mathcal{C}_2 \subseteq \nabla\mathcal{B}_2$. Let $b_i = |\mathcal{B}_i|$, $c_i = |\mathcal{C}_i|$, for $i = 1, 2$. Since $\overline{\nabla}(\mathcal{A}_j) \cap \mathcal{A}_j = \emptyset$, then $\mathcal{C}_1 \cap \mathcal{B}_2 = \emptyset$, moreover since both \mathcal{B}_2 and \mathcal{C}_1 are $\frac{n}{2}$ -subsets of X we get $b_2 + c_1 \leq \binom{n}{\frac{n}{2}}$. Furthermore, since \mathcal{A}_j is an antichain we also get $b_1 + b_2 \leq \binom{n}{\frac{n}{2}}$. Hence, since $2|\mathcal{A}| + |\overline{\nabla}(\mathcal{A})| = 2(b_1 + b_2) + (c_1 + c_2)$, we have: $2|\mathcal{A}| + |\overline{\nabla}(\mathcal{A})| \leq 2\binom{n}{\frac{n}{2}} + (b_1 + c_2)$. Thus to prove the theorem, it is enough to show $b_1 + c_2 \leq \binom{n}{\frac{n}{2}+1}$. Note that \mathcal{B}_1 is formed by $\frac{n}{2} - 1$ -subsets of X , while the elements of \mathcal{C}_2 are $\frac{n}{2} + 1$ -subsets. We claim that $\mathcal{B}_1 \cup \mathcal{C}_2$ is an antichain. Indeed, if there is a $z \in \mathcal{B}_1$ and $z' \in \mathcal{C}_2$ with $z \subsetneq z'$, then, since $|z'| = \frac{n}{2} + 1$ and $|z| = \frac{n}{2} - 1$ there is a $a \in X$ with $z \cup \{a\} \subsetneq z'$. However $z \cup \{a\} \in \nabla\mathcal{B}_1 = \mathcal{C}_1$ and $z' \in \mathcal{C}_2$ contradict the fact that $\overline{\nabla}\mathcal{A}_j$ is an antichain. Therefore $\mathcal{B}_1 \cup \mathcal{C}_2$ is an antichain. Let A_1, \dots, A_ℓ be a symmetric chains decomposition of the set of subsets of X (see [5, Section 3.2]). We define the map $\varphi : \mathcal{B}_1 \rightarrow 2^X$ which associates to each $z \in \mathcal{B}_1$ with $z \in A_i$, for some $i \in \{1, \dots, \ell\}$, the “specular” set $\varphi(z)$ in A_i with $|z| + |\varphi(z)| = n$. Note that φ is clearly injective, furthermore it sends $\frac{n}{2} - 1$ -subsets into $\frac{n}{2} + 1$ -subsets.

Thus, to prove $b_1 + c_2 \leq \binom{n}{\frac{n}{2}+1}$, it is enough to show $\varphi(\mathcal{B}_1) \cap \mathcal{C}_2 = \emptyset$. Indeed, suppose, contrary to our claim, that there is $z' \in \varphi(\mathcal{B}_1) \cap \mathcal{C}_2$, then we can find a $z \in \mathcal{B}_1$ with $\varphi(z) = z'$. Since z, z' belong to the same symmetric chain, we get $z \subsetneq z'$ which contradicts the fact that $\mathcal{B}_1 \cup \mathcal{C}_2$ is an antichain and this concludes the proof of the theorem. \square

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